Math Study Guide
U.1: Rational Expressions, Exponents, Factoring, Inequalities

1.1

Exponent Rules

Rule	Description	Example
Product	$a^m \times a^n = a^{m+n}$	$4^2 \times 4^5 = 4^7$
Quotient	$a^m \div a^n = a^{m-n}$	$5^4 \div 5^2 = 5^2$
Power of a power	$(a^m)^n = a^{m\times n}$	$(3^2)^4 = 3^8$
Power of a product	$(xy)^a = x^a y^a$	$(2 \times 3)^2 = 2^2 \times 3^2$
Power of a quotient	$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}, b \neq 0$	$\left(\frac{3}{4}\right)^5 = \frac{3^5}{4^5}$
Zero as an exponent	$a^0 = 1$	$7^0 = 1$
Negative exponents **1.2**	$a^{-m} = \frac{1}{a^m}, a \neq 0$	$9^{-2} = \frac{1}{9^2}$
Rational Exponents	$a^{\frac{m}{n}} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$ $a^{power/root} = a^{m/n}$ (alphabetical!)	$27^{\frac{4}{3}} = \sqrt[3]{27^4} = \left(\sqrt[3]{27}\right)^4$
Negative Rational Exponents	$x^{-m/n} = 1/x^{m/n} = 1/\sqrt[n]{x^m}$	$(25/4)^{-3/2} = (4/25)^{3/2} =$ $(4^{3/2})/(25^{3/2}) = \sqrt{4^3}/\sqrt{25^3} = 8/125$

Rational = Fraction

Radical = Root

1.3

Solving Exponential Equations

e.g. Solve for x.

$9^{x-2} - 8 = 73$

Add 8 to both sides. Simplify.

$9^{x-2} = 73 + 8$

$9^{x-2} = 81$

Note LS and RS are powers of 9, so rewrite them as powers using the same base.

$9^{x-2} = 9^2$

$x - 2 = 2$

$x = 2 + 2$

$x = 4$

When the bases are the same, equate the exponents. Solve for x.

Don't forget to check your solution!

$LS = 9^{x-2} - 8 \quad RS = 73$

$= 9^{4-2} - 8$

$= 81 - 8$

$= 73 = RS \quad x = 4$ checks

Exponential Growth and Decay

Population growth and radioactive decay can be modelled using exponential functions.

Decay: $A_L = A_0\left(\frac{1}{2}\right)^{\frac{t}{h}}$ A_0 - initial amount t – time elapsed h – half-life

A_L - amount Left

Factoring Review

Factoring Polynomials

To **expand** means to write a product of polynomials as a sum or a difference of terms.

To **factor** means to write a sum or a difference of terms as a product of polynomials.

Factoring is the inverse operation of expanding.

Expanding

$(2x+3)(3x-7) = 6x^2 - 5x - 21$

Factoring

Product of polynomials

Sum or difference of terms

Types of factoring:

Common Factors: factors that are common among each term.

e.g. Factor,

$$35m^3n^3 - 21m^2n^2 + 56m^2n$$
$$= 7m^2n(5mn^2 - 3n + 8)$$

Each term is divisible by $7m^2n$.

Factor by grouping: group terms to help in the factoring process.

e.g. Factor,

$A: 4mx + ny - 4nx - my$
$= 4mx - 4nx + ny - my$
$= 4x(m-n) + y(n-m)$
$= 4x(m-n) - y(m-n)$
$= (4x - y)(m-n)$

Group $4mx - 4nx$ and $ny - my$, factor each group

Recall $n - m = -(m - n)$

Common factor

$B: 1 + 6x + 9x^2 - 4y^2$
$= (1+3x)^2 - 4y^2$
$= [(1+3x)+2y][(1+3x)-2y]$
$= (1+3x+2y)(1+3x-2y)$

$1+6x+9x^2$ is a perfect square trinomial

Difference of squares

Factoring $ax^2 + bx + c$

Find the product of *ac*. Find two numbers that multiply to *ac* and add to *b*.

e.g. Factor,

$A: y^2 + 9y + 14$
$= y^2 + 7y + 2y + 14$
$= y(y+7) + 2(y+7)$
$= (y+2)(y+7)$

Product = 14 = 2(7)
Sum = 9 = 2 + 7

$B: 3x^2 - 7xy - 6y^2$
$= 3x^2 - 9xy + 2xy - 6y^2$
$= 3x(x-3y) + 2y(x-3y)$
$= (3x+2y)(x-3y)$

Product = 3(–6) = –18 = –9(2)
Sum = – 7 = –9 + 2
Decompose middle term $-7xy$ into $-9xy + 2xy$.
Factor by grouping.

Sometimes polynomials can be factored using **special patterns**.

Perfect square trinomial $a^2 + 2ab + b^2 = (a+b)(a+b)$ or $a^2 - 2ab + b^2 = (a-b)(a-b)$

e.g. Factor,

$A: 4p^2 + 12p + 9$
$= (2p+3)^2$

$B: 100x^2 - 80xy + 16y^2$
$= 4(25x^2 - 20xy + 4y^2)$
$= 4(5x-2y)(5x-2y)$

Difference of squares $a^2 - b^2 = (a+b)(a-b)$

e.g. Factor, $9x^2 - 4y^2 = (3x+2y)(3x-2y)$

1.4

+, - , X, Polynomials

A polynomial is an algebraic expression with real coefficients and non-negative integer exponents.

A polynomial with 1 term is called a monomial, $7x$.

A polynomial with 2 terms is called a binomial, $3x^2 - 9$.

A polynomial with 3 terms is called a trinomial, $3x^2 + 7x - 9$.

The degree of the polynomial is determined by the value of the highest exponent of the variable in the polynomial.

e.g. $3x^2 + 7x - 9$, degree is 2.

For polynomials with one variable, if the degree is 0, then it is called a constant.

If the degree is 1, then it is called linear.

If the degree is 2, then it is called quadratic.

If the degree is 3, then it is called cubic.

We can add and subtract polynomials by collecting like terms.

e.g. Simplify.

The negative in front of the brackets applies to every term inside the brackets. That is, you multiply each term by –1.

$$\left(5x^4 - x^2 - 2\right) - \left(x^4 - 2x^3 + 3x^2 - 5\right)$$
$$= 5x^4 - x^2 - 2 - x^4 + 2x^3 - 3x^2 + 5$$
$$= 5x^4 - x^4 + 2x^3 - x^2 - 3x^2 - 2 + 5$$
$$= 4x^4 + 2x^3 - 4x^2 + 3$$

To multiply polynomials, multiply each term in the first polynomial by each term in the second.

e.g. Expand and simplify.

$$\left(x^2 + 4\right)\left(x^2 - 2x + 3\right)$$
$$= x^4 - 2x^3 + 3x^2 + 4x^2 - 8x + 12$$
$$= x^4 - 2x^3 + 7x^2 - 8x + 12$$

1.5

Rational Expressions

For polynomials F and G, a rational expression is formed when $\frac{F}{G}$, $G \neq 0$.

e.g. $\frac{3x+7}{21x^2+14x+9}$

Simplifying Rational Expressions

e.g. Simplify and state the restrictions.

$$\frac{m^2-9}{m^2+6m+9} = \frac{(m+3)(m-3)}{(m+3)(m+3)}$$

Factor the numerator and denominator. Note the restrictions. $m \neq -3$

$$= \frac{\cancel{(m+3)}(m-3)}{\cancel{(m+3)}(m+3)}$$

Simplify.

$$= \frac{m-3}{m+3},\ m \neq -3$$

State the restrictions.

1.6

Multiplying and Dividing Rational Expressions

e.g. Simplify and state the restrictions.

$$A: \frac{x^2+7x}{x^2-1} \times \frac{x^2+3x+2}{x^2+14x+49}$$

$$= \frac{x(x+7)}{(x+1)(x-1)} \times \frac{(x+1)(x+2)}{(x+7)(x+7)}$$

Factor. Note restrictions.

$$= \frac{x\cancel{(x+7)}}{\cancel{(x+1)}(x-1)} \times \frac{\cancel{(x+1)}(x+2)}{\cancel{(x+7)}(x+7)}$$

Simplify.

$$= \frac{x(x+2)}{(x-1)(x+7)},\ x \neq \pm 1, -7$$

State restrictions.

$$B: \frac{x^2-9}{x^2+5x+4} \div \frac{x^2-4x+3}{x^2+5x+4}$$

$$= \frac{(x+3)(x-3)}{(x+4)(x+1)} \div \frac{(x-1)(x-3)}{(x+4)(x+1)}$$

Factor. Note restrictions.

$$= \frac{(x+3)(x-3)}{(x+4)(x+1)} \times \frac{(x+4)(x+1)}{(x-1)(x-3)}$$

Invert and multiply. Note any new restrictions.

$$= \frac{(x+3)\cancel{(x-3)}}{\cancel{(x+4)(x+1)}} \times \frac{\cancel{(x+4)(x+1)}}{(x-1)\cancel{(x-3)}}$$

Simplify.

$$= \frac{(x+3)}{(x-1)},\ x \neq -4, \pm 1, 3$$

State restrictions.

1.7 and 1.8

Adding and Subtracting Rational Expressions

Note that after addition or subtraction it may be possible to factor the numerator and simplify the expression further. Always reduce the answer to lowest terms.

e.g. Simplify and state the restrictions.

$$A: \frac{3}{x^2-4}+\frac{5}{x+2}$$

$$= \frac{3}{(x-2)(x+2)}+\frac{5}{x+2}$$

Factor.
Note restrictions.
Simplify if possible.

$$= \frac{3}{(x-2)(x+2)}+\frac{5(x-2)}{(x+2)(x-2)}$$

Find LCD.
Write all terms using LCD.

$$= \frac{3+5x-10}{(x+2)(x-2)}$$

Add.

$$= \frac{5x-7}{(x+2)(x-2)},\ x \neq \pm 2$$

State restrictions.

$$B: \frac{2}{x^2-xy}-\frac{3}{xy-y^2}$$

Factor.
Note restrictions.
Simplify if possible.

$$= \frac{2}{x(x-y)}-\frac{3}{y(x-y)}$$

Find LCD.
Write all terms using LCD.

$$= \frac{2y}{xy(x-y)}-\frac{3x}{xy(x-y)}$$

$$= \frac{2y-3x}{xy(x-y)},\ x \neq 0, y,\ \ y \neq 0$$

Subtract.
State restrictions.

1.9

Linear inequalities are also called first degree inequalities
E.g. $4x > 20$ is an inequality of the first degree, which is often called a linear inequality.
Recall that:

- the same number can be subtracted from both sides of an inequality
- the same number can be added to both sides of an inequality
- both sides of an inequality can be multiplied (or divided) by the same positive number
- if an inequality is multiplied (or divided) by the same negative number, then:

$\geq$ becomes $\leq$
$>$ becomes $<$
$\leq$ becomes $\geq$
$<$ becomes $>$

EX. Solve the inequality $x + 32 > 36$ for x.

$x + 32 > 36$ {Subtract 32 from each side}
$\therefore\ x + 32 - 32 > 36 - 32$
$x + 0 > 4$
$x > 4$

Answer = x > 4

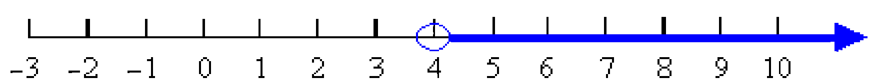


Figure 1 Note: On a number line, a closed dot means including the end point and an opened dot means excluding the end point

U.2: Quadratics and Radicals

2.4

Radicals

e.g. $\sqrt[n]{a}$, $\sqrt{\ }$ is called the radical sign, n is the index of the radical, and a is called the radicand.
$\sqrt{3}$ is said to be a radical of order 2. $\sqrt[3]{8}$ is a radical of order 3.

Like radicals:

$\sqrt{5}, 2\sqrt{5}, -3\sqrt{5}$

Same order, like radicands

Entire radicals:

$\sqrt{8}, \sqrt{16}, \sqrt{29}$

Mixed radicals:

$4\sqrt{2}, 2\sqrt{3}, 5\sqrt{7}$

Unlike radicals: $\sqrt{5}, \sqrt[3]{5}, \sqrt{3}$

Different order

Different radicands

A radical in **simplest form** meets the following conditions:

For a radical of order *n*, the radicand has no factor that is the *n*th power of an integer. The radicand contains no fractions.

$$\sqrt{\frac{3}{2}} = \sqrt{\frac{3}{2}\times\frac{2}{2}}$$
$$= \sqrt{\frac{6}{2^2}}$$
$$= \frac{\sqrt{6}}{\sqrt{2^2}}$$
$$= \frac{\sqrt{6}}{2}$$

Simplest form

The radicand contains no factors with negative exponents.

$$\sqrt{a^{-1}} = \sqrt{\frac{1}{a}}$$

Not simplest form

$$= \sqrt{\frac{1}{a}\times\frac{a}{a}}$$
$$= \sqrt{\frac{a}{a^2}}$$
$$= \frac{\sqrt{a}}{a}$$

Simplest form

The index of a radical must be as small as possible.

$$\sqrt[4]{3^2} = \sqrt{\sqrt{3^2}}$$
$$= \sqrt{3}$$

Simplest form

Addition and Subtraction of Radicals

To add or subtract radicals, you add or subtract the coefficients of each radical.

e.g. Simplify.

$$2\sqrt{12} - 5\sqrt{27} + 3\sqrt{40} = 2\sqrt{4\times 3} - \ldots$$

Express each radical in simplest form.

$$= 2(2\sqrt{3}) - 5(3\sqrt{3}) + 3(2\sqrt{10})$$
$$= 4\sqrt{3} - 15\sqrt{3} + 6\sqrt{10}$$

Collect like radicals. Add and subtract.

$$= -11\sqrt{3} + 6\sqrt{10}$$

$$\sqrt{8} = \sqrt{4\times 2}$$
$$= \sqrt{2^2\times 2}$$
$$= 2\sqrt{2}$$

Multiplying Radicals

$$\sqrt{a}\times\sqrt{b} = \sqrt{ab},\ a \ge 0,\ b \ge 0$$

e.g. Simplify.

$$(\sqrt{2} + 2\sqrt{3})(\sqrt{2} - 3\sqrt{3}) = (\sqrt{2})(\sqrt{2}) - (\sqrt{2})(3\sqrt{3}) + (2\sqrt{3})(\sqrt{2}) - (2\sqrt{3})(3\sqrt{3})$$

Use the distributive property to expand

$$= 2 - 3\sqrt{6} + 2\sqrt{6} - 6(3)$$

Multiply coefficients together. Multiply radicands together.

$$= 2 - 18 - 3\sqrt{6} + 2\sqrt{6}$$
$$= -16 - \sqrt{6}$$

Collect like terms. Express in simplest form.

Conjugates

$(a\sqrt{b} + c\sqrt{d})$ and $(a\sqrt{b} - c\sqrt{d})$ are called conjugates.

Opposite signs

Same terms

Same terms

When conjugates are multiplied the result is a rational expression (no radicals).

e.g. Find the product.

$$(\sqrt{5} + 3\sqrt{2})(\sqrt{5} - 3\sqrt{2}) = (\sqrt{5})^2 - (3\sqrt{2})^2$$
$$= 5 - 9(2)$$
$$= 5 - 18$$
$$= -13$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}},\ a,b \in \mathbf{R},\ a \geq 0, b \geq 0$$

Dividing Radicals

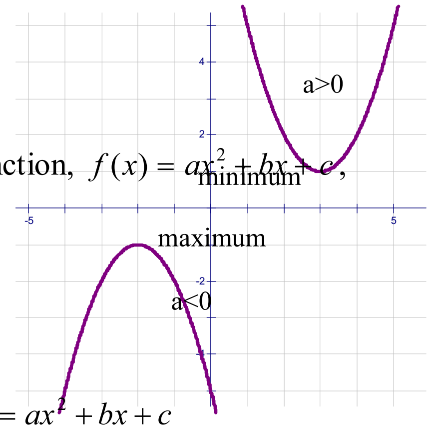
e.g. Simplify.

$$\frac{2\sqrt{10}+3\sqrt{30}}{\sqrt{5}} = \frac{2\sqrt{10}}{\sqrt{5}} + \frac{3\sqrt{30}}{\sqrt{5}}$$
$$= 2\sqrt{\frac{10}{5}} + 3\sqrt{\frac{30}{5}}$$
$$= 2\sqrt{2} + 3\sqrt{6}$$

<u>2.2 and 2.3</u>

<u>Quadratic Functions</u>

The graph of the quadratic function, $f(x) = ax^2 + bx + c$, is a parabola.



When $a > 0$ the parabola opens up. When $a < 0$ the parabola opens down.

Vertex Form: $f(x) = a(x-h)^2 + k$

The vertex is (h, k). The maximum or minimum value is k.

The axis of symmetry is $y = h$.

Factored Form: $f(x) = a(x-p)(x-q)$

The zeroes are $x = p$ and $x = q$.

Standard Form: $f(x) = ax^2 + bx + c$

The y-intercept is c.

Complete the square to change the standard form to vertex form.

e.g.

$f(x) = -2x^2 - 12x + 7$ — Factor the coefficient of x^2 form the terms with x^2 and x.

$f(x) = -2(x^2 + 6x) + 7$ — Divide the coefficient of x by 2. Square this number. Add and subtract it.

$f(x) = -2(x^2 + 6x + 3^2 - 3^2) + 7$ — Bring the last term inside the bracket outside the brackets.

$f(x) = -2(x^2 + 6x + 3^2) - 2(-3^2) + 7$ — Factor the perfect square trinomial inside the brackets.

$f(x) = -2(x+3)^2 - 2(-9) + 7$ — Simplify.

$f(x) = -2(x+3)^2 + 25$

Maximum and Minimum Values

Vertex form, maximum/minimum value is k.

Factored form:

e.g. Determine the maximum or minimum value of $f(x) = (x-1)(x-7)$.

The zeroes of $f(x)$ are equidistant from the axis of symmetry. The zeroes are $x = 1$ and $x = 7$.

$x = \frac{1+7}{2}$

$x = 4$

The axis of symmetry is $x = 4$. The axis of symmetry passes through the vertex. The x-coordinate of the vertex is 4. To find the y-coordinate of the vertex, evaluate $f(4)$.

$f(4) = (4-1)(4-7)$

$f(4) = 3(-3)$

$f(4) = -9$

The vertex is $(4, -9)$. Because a is positive ($a = 1$), the graph opens up. The minimum value is –9.

Standard form:

e.g. Determine the maximum or minimum value of $f(x) = -2x^2 - 10x + 10$ without completing the square.

$g(x) = -2x^2 - 10x$ is a vertical translation of $f(x) = -2x^2 - 10x + 10$ with y-intercept of 0.

$g(x) = -2x(x+5)$ $\quad x = 0, -5$ are the zeroes.

$x = \frac{0-5}{2} = -2.5$ $\quad x = -2.5$ is the x-coordinate of vertex.

$f(-2.5) = -2(-2.5)2 - 10(-2.5) + 10$

$f(-2.5) = 22.5$

The y-coordinate of vertex is 22.5. It is a maximum

Factor $g(x) = -2x^2 - 10x$ to determine zeroes, then find the axis of symmetry. Both $f(x)$ and $g(x)$ will have the same x-coordinates for the vertex. To find the y-coordinate for *f(x)* simply evaluate *f(x)* using the same x-coordinate.

because the graph opens down.

Zeroes

To determine the number of zeroes of a quadratic function consider the form of the function.

Vertex form: If a and k have opposite signs there are 2 zeroes (2 roots).
If a and k have the same sign there are no zeroes (0 roots).
If $k = 0$ there is one zero (1 root).

Factored form: $f(x) = a(x-p)(x-q)$ $\rightarrow$ 2 zeroes. The zeroes are $x = p$ and $x = q$.
$f(x) = a(x-p)^2$ $\rightarrow$ 1 zero. The zero is $x = p$.

Standard form: Check discriminant. $D = b^2 - 4ac$
If $D < 0$ there are no zeroes.
If $D = 0$ there is 1 zero.
If $D > 0$ there are 2 zeroes.

To determine the zeroes of from the standard form use the **quadratic formula**.

For , $ax^2 + bx + c = 0$ use $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve for x.

U.3: Functions

3.1

Functions

A **relation** is a set of any ordered pairs. Relations can be described using:

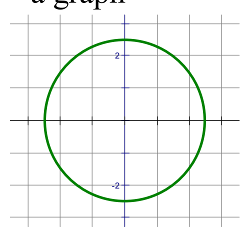
an equation
$y = 3x^2 - 7$

in words
"output is three more than input"

a set of ordered pairs
$\{(1,2),(0,3),(4,8)\}$

function notation
$f(x) = x^2 - 3x$

a graph



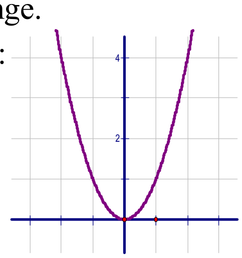
a table

x	y
1	2
2	3
3	4
4	3

The **domain** of a relation is the set of possible input values (x values).
The **range** is the set of possible output values (y values).
e.g. State the domain and range.

A: $\{(1,2),(0,3),(4,8)\}$
Domain = $\{0, 1, 4\}$
Range = $\{2, 3, 8\}$

B:



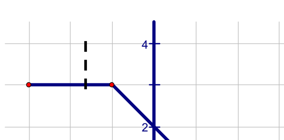
Looking at the graph we can see that y does not go below 0. Thus,
Domain = **R**
Range = $\{y \mid y \geq 0, y \in R\}$

C: $y = \sqrt{x-5}$
What value of x will make $x - 5 = 0$? $x = 5$
The radicand cannot be less than zero, so
Domain = $\{x \mid x \geq 5, x \in \mathbf{R}\}$
Range = $\{y \mid y \geq 0, y \in \mathbf{R}\}$

A **function** is a special type of relation in which no two ordered pairs have the same x value.
$y = x - 7$ and $y = x^2 + 15$ are examples of functions. $y = \pm\sqrt{x}$ is not a function because for every value of x there are two values of y.

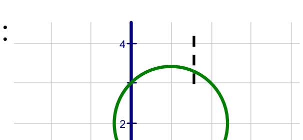
The **vertical line test** is used to determine if a graph of a relation is a function. If a vertical line can be passed along the entire length of the graph and it never touches more than one point at a time, then the relation is a function.

e.g. A:



This passes the vertical line test, so

B:



The line passes through more than one point, so this relation fails the vertical

3.2

More Function Notes

** = asymptote = a line that a curve approaches more and more closely. The line x=0 is a ~ of the graph y =1/x. the function y = 1/x is not defined when x = 0 (rmr restrictions? Denominator can never = 0, its an illegal operation). Therefore the graph for this is said to be *discontinuous* at x = 0.

** invariant points = points that are unaltered by a transformation, i.e. in -f(x), when it reflects on the x axis, any points that already lie on the x axis will be UNAFFECTED. Same with f(-x) except for y axis. And for -f(-x), only the origin (0,0) will be unaffected

Reciprocal functions

The reciprocal function of a function, f, is defined as $\frac{1}{f}$. To help you graph $y=\frac{1}{f(x)}$, you should use the following:

The vertical asymptotes of $y=\frac{1}{f(x)}$ will occur where $f(x)=0$

As $f(x)$ increases, $\frac{1}{f(x)}$ decreases. As $f(x)$ decreases, $\frac{1}{f(x)}$ increases.

For $f(x)>0$, $\frac{1}{f(x)}>0$. For $f(x)<0$, $\frac{1}{f(x)}<0$.

The graph of $y=\frac{1}{f(x)}$ always passes through the points where $f(x)=1$ or $f(x)=-1$.

You may find it helpful to sketch the graph of $y=f(x)$ first, before you graph the reciprocal.

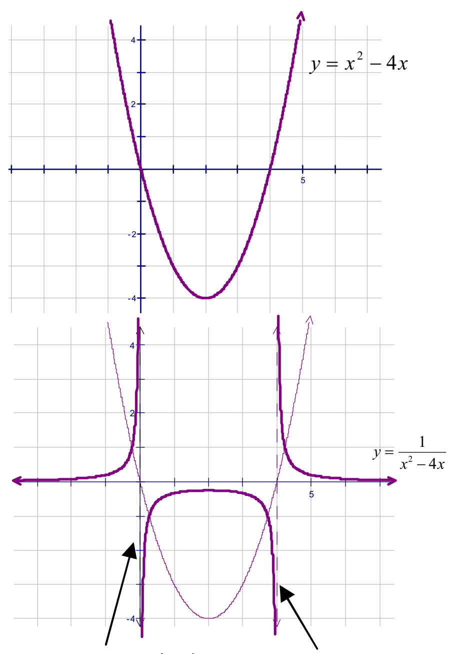
e.g. Sketch the graph of $y=\frac{1}{x^2-4x}$.

Look at the function $f(x)=x^2-4x$.

Factor it. $f(x)=x(x-4)$.

The zeroes are $x = 0$, and $x = 4$. The vertical asymptotes will be at $x = 0$, and $x = 4$.

You could sketch the graph of $f(x)=x^2-4x$ to see where the function increases and decreases, where $f(x)=1$ or –1. Use the information above to help you sketch the reciprocal.



Vertical asymptotes

3.3

TRANSLATIONS OF FUNCTIONS

Vertical: y = f(x) + k   if k is positive >0, moves up. if negative, moves down

Horizontal: y = f(x-h)   if h is positive >0, moves right (WATCH THE NEGATIVE SIGN IN FRONT OF THE H THOUGH, can trick you!), and if it is negative <0, moves left

*Note: this doesn't stretch, only translates!

3.4

REFLECTIONS OF FUNCTIONS

Y = -f(x) the negative in front of the function causes a reflection in the **x** axis (becomes (x,-y))

Y = f(-x) the negative in side of the function in front of the x causes a reflection in the **y** axis (becomes (-x,y))

*=A function and its reflection are still the same shape: still CONGRUENT

3.5

Inverse Functions

The inverse, f^{-1}, of a relation, f, maps each output of the original relation back onto the corresponding input value. The domain of the inverse is the range of the function, and the range of the inverse is the domain of the function. That is, if $(a,b)\in f$, then $(b,a)\in f^{-1}$. The graph of $y=f^{-1}(x)$ is the reflection of the graph $y=f(x)$ in the line $y=x$.

e.g. Given $f(x)=\dfrac{3x-1}{5}$.

Evaluate $f(-3)$.

$$f(-3)=\frac{3(-3)-1}{5}$$
$$f(-3)=\frac{-9-1}{5}$$
$$f(-3)=\frac{-10}{5}$$
$$f(-3)=-2$$

Replace all x's with –3. Evaluate.

Evaluate $3f(2)+1$

$$3f(2)+1=3\left[\frac{3(2)-1}{5}\right]+1$$
$$=3\left[\frac{6-1}{5}\right]+1$$
$$=3\left(\frac{5}{5}\right)+1$$
$$=3(1)+1$$
$$3f(2)+1=4$$

You want to find the value of the expression $3f(2)+1$. You are <u>not</u> solving for $f(2)$.

Determine $f^{-1}(x)$.

$$y=\frac{3x-1}{5}$$
$$x=\frac{3y-1}{5}$$
$$5x=3y-1$$
$$3y=5x+1$$
$$y=\frac{5x+1}{3}$$
$$\therefore\ f^{-1}(x)=\frac{5x+1}{3}$$

Rewrite $f(x)$ as $y=\dfrac{3x-1}{5}$

Interchange x and y. Solve for y.

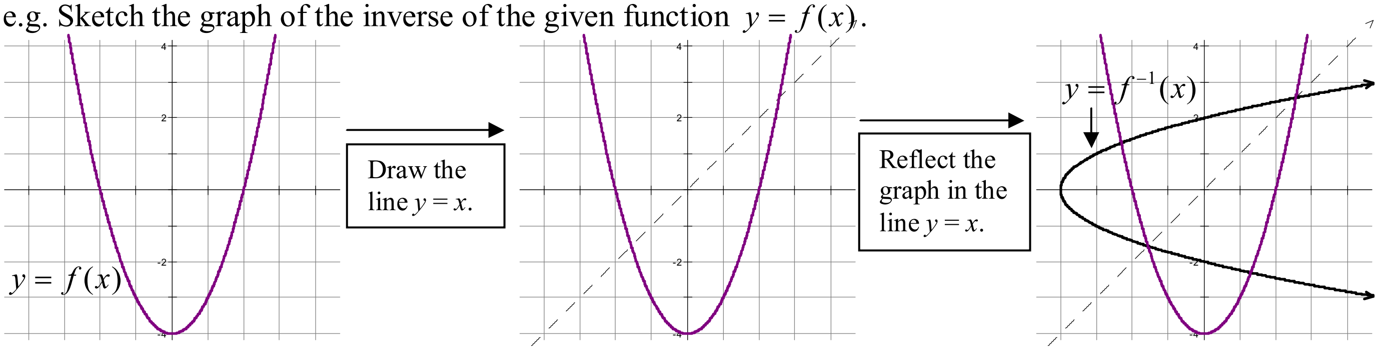
Evaluate $f^{-1}(2)$

$$f^{-1}(x)=\frac{5x+1}{3}$$
$$f^{-1}(2)=\frac{5(2)+1}{3}$$
$$=\frac{10+1}{3}$$
$$f^{-1}(2)=\frac{11}{3}$$

If you have not already determined $f^{-1}(x)$ do so.

Using $f^{-1}(x)$, replace all x's with 2. Evaluate.

e.g. Sketch the graph of the inverse of the given function $y=f(x)$.



The inverse of a function is not necessarily going to be a function. If you would like the inverse to also be a function, you may have to restrict the domain or range of the original function. For the example above, the inverse will only be a function if we restrict the domain to $\{x \mid x \geq 0, x \in \mathbf{R}\}$ or $\{x \mid x \leq 0, x \in \mathbf{R}\}$.

3.7

Transformations of Functions

Mathematical Form:	*Effect:*
$y = f(x) + k$	• if $k > 0$, we move up k units • " $k < 0$, " " down k units
$y = f(x - h)$	• if $h > 0$ we move right h units • " $h < 0$, " " left h units * Watch for negative sign tho!
$y = -f(x)$	reflection in x axis
$y = f(-x)$	reflection in y axis
$y = af(x)$	• if $a > 1$, vertical stretch by a factor of a • if $a < 1$, vertical compression by a factor of a
$y = f(kx)$	• if $k > 1$, horizontal compression by a factor of $\frac{1}{k}$ • if $0 < k < 1$, horizontal stretch by a factor of $\frac{1}{k}$

Order of Transformations

When combining transformations, to simplify the procedure and give the desired results, ***perform the transformations in the following order:***

1) stretches/compressions first
2) reflections
3) left, right, up, down

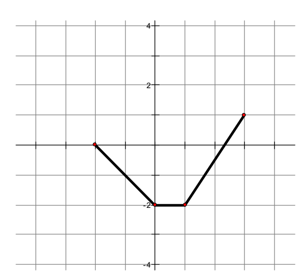
To graph $y = af[k(x-p)] + q$ from the graph $y = f(x)$ consider:

a – determines the vertical stretch. The graph $y = f(x)$ is stretched vertically by a factor of *a*. If $a < 0$ then the graph is reflected in the *x*-axis, as well.

k – determines the horizontal stretch. The graph $y = f(x)$ is stretched horizontally by a factor of $\frac{1}{k}$. If $k < 0$ then the graph is also reflected in the *y*-axis.

p – determines the horizontal translation. If $p > 0$ the graph shifts to the right by *p* units. If $p < 0$ then the graph shifts left by *p* units.

q – determines the vertical translation. If $q > 0$ the graph shifts up by *q* units. If $q < 0$ then the graph shifts down by *q* units.



When applying transformations to a graph the stretches and reflections should be applied before any translations.

e.g. The graph of $y = f(x)$ is transformed into $y = 3f(2x-4)$. Describe the transformations.

First, factor inside the brackets to determine the values of *k* and *p*.

$y = 3f(2(x-2))$

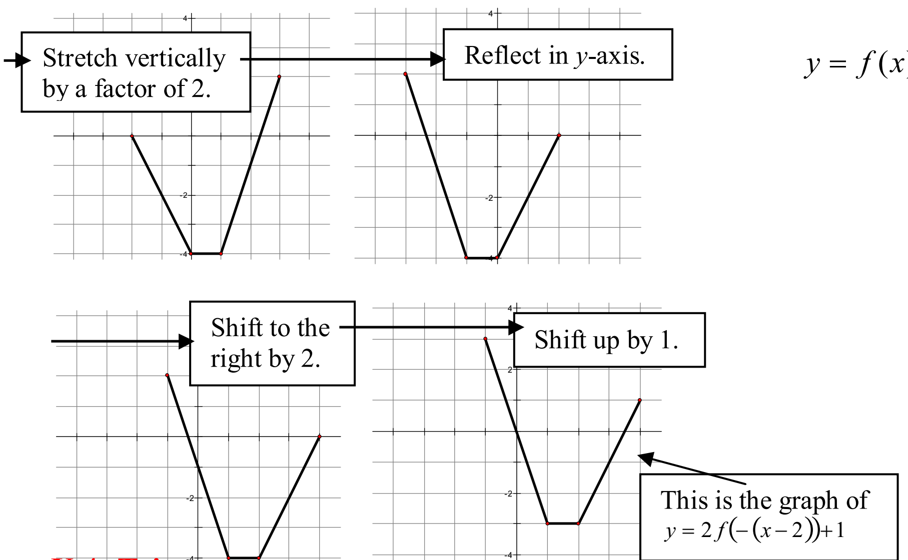
$a = 3, k = 2, p = 2$

There is a vertical stretch of 3.

A horizontal stretch of $\frac{1}{2}$.

The graph will be shifted 2 units to the right.

e.g. Given the graph of $y = f(x)$ sketch the graph of $y = 2f(-(x-2))+1$



U.4: Trig

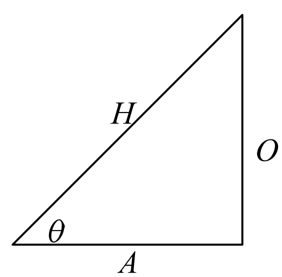
4.1

Trigonometry

Given a right angle triangle we can use the following ratios

Primary Trigonometric Ratios

$\sin\theta = \frac{O}{H}$ $\cos\theta = \frac{A}{H}$ $\tan\theta = \frac{O}{A}$

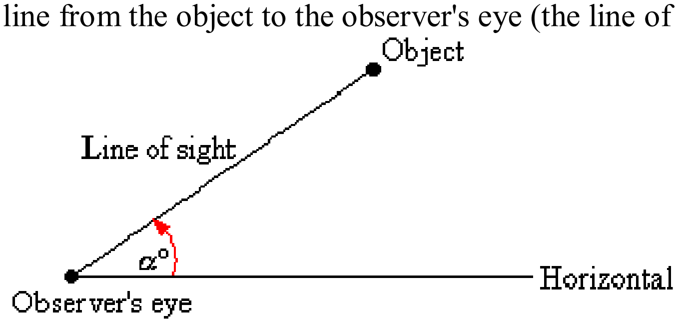


Reciprocal Trigonometric Ratios

$\csc\theta = \frac{r}{y} = \frac{1}{\sin\theta}$ $\sec\theta = \frac{r}{x} = \frac{1}{\cos\theta}$ $\cot\theta = \frac{x}{y} = \frac{1}{\tan\theta}$

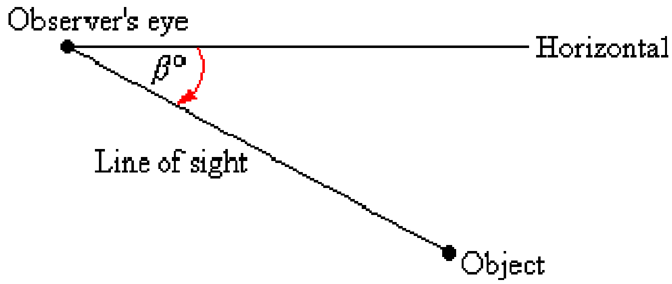
Angles of Depression/Elevation

he **angle of elevation** of an object as seen by an observer is the angle between the horizontal and the line from the object to the observer's eye (the line of sight).



The angle of elevation of the object from the observer is $\alpha°$.

If the object is below the level of the observer, then the angle between the horizontal and the observer's line of sight is called the **angle of depression**.



The angle of depression of the object from the observer is $\beta°$.

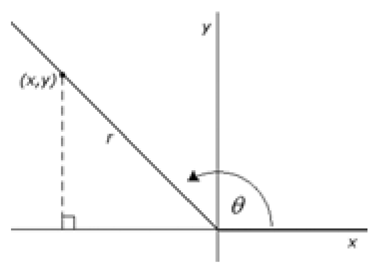
NOTE: Math teachers expect you not to use rounded values if you don't have to IN A TRIG RATIO

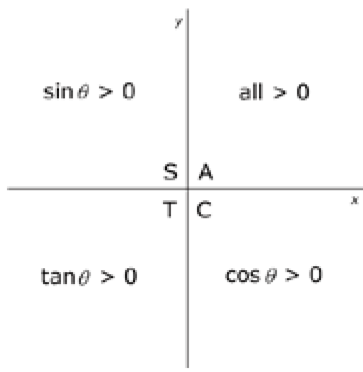
4.2

For angles that are *obtuse* (angle is greater than 90°) or negative, we use the following trigonometric ratios. The x and y variables are the values of the x and y coordinates, respectively. The r variable represents the distance from the origin, to the point (x,y). This value can be found using the Pythagorean theorem.

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

$$\csc\theta = \frac{r}{y} \qquad \sec\theta = \frac{r}{x} \qquad \cot\theta = \frac{x}{y}$$



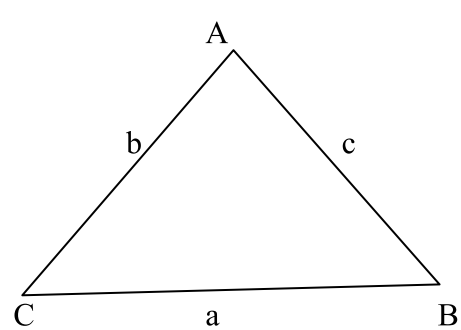


When negative or obtuse angles are used in trigonometric functions, they will sometimes produce negative values. The CAST graph to the left will help you to remember the signs of trigonometric functions for different angles. The functions will be negative in all quadrants except those that indicate that the function is positive. For example, When the angle is between 0° and 90° (0 and pi/2 radians), the line r is in the A quadrant. All functions will be positive in this region. When the angle is between 90° and 180° (pi/2 and pi radians), the line is in the S quadrant. This means that only the sine function is positive. All other functions will be negative.

***CAST**

4.3

Trigonometry of Oblique Triangles
Sine Law



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

When you know 2 angles and a side OR 2 sides and an angle OPPOSITE one of these sides

Cosine Law

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

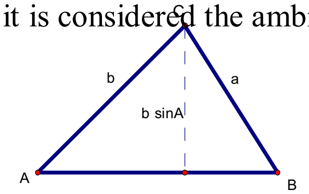
(and etc. for b and c)

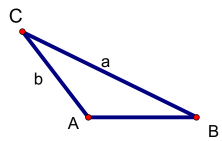
When you know all 3 sides OR 2 sides and any contained angle□AMBIGUOUS CASE!

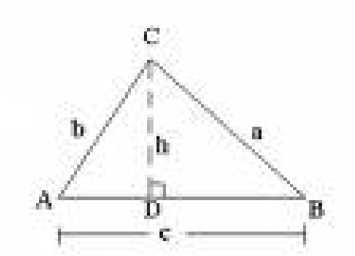
4.4

When you know sides and any contained angle it is considered the ambiguous case.

Angle	Conditions	# of Triangles
$\angle A < 90^\circ$	$a < b\sin A$	0
	$a = b\sin A$	1
	$a > b\sin A$	2
$\angle A > 90^\circ$	$a \leq b$	0
	$a > b$	1







**height of a triangle = bsinA

5.1

Angles can be expressed in degrees or radians. To convert a measurement from radians to degrees (or vice versa) we use the following relationship:

$$\pi\, rad = 180^\circ$$

This relationship gives the following two equations:

$$1\, rad = \left(\frac{180}{\pi}\right)^\circ \qquad 1^\circ = \frac{\pi}{180}\, rad$$

Note: By convention, most angles are expressed in radian measure, unless otherwise stated.

Examples:

I) Convert $\frac{\pi}{6}$ radians to degrees.

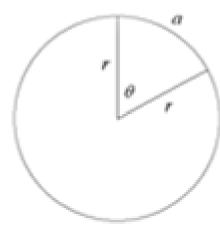
$$\begin{aligned}\frac{\pi}{6}\text{ rad} &= \frac{\pi}{6}(1\text{ rad})\\ &= \frac{\pi}{6}\left(\frac{180}{\pi}\right)^{\circ}\\ &= \left(\frac{180\pi}{6\pi}\right)^{\circ}\\ &= \left(\frac{180}{6}\right)^{\circ}\\ &= 30^{\circ}\end{aligned}$$

II) Convert -210° to radians :

$$\begin{aligned}-210^{\circ} &= -210(1^{\circ})\\ &= -210\left(\frac{\pi}{180}\text{ rad}\right)\\ &= \frac{-210\pi}{180}\text{ rad}\\ &= \frac{(30)(-7\pi)}{(30)(6)}\text{ rad}\\ &= \frac{-7\pi}{6}\text{ rad}\end{aligned}$$

Arc Length

The arc length formula defines the relationship between arc length a, radius r and the angle θ (in radians).



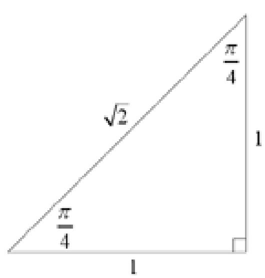
$$a = r\theta$$

Note: Make sure that your angles are measured in radians. The arc length formula does not hold for angles measured in degrees. Use the conversion relationship above to convert your angles from degrees to radians.

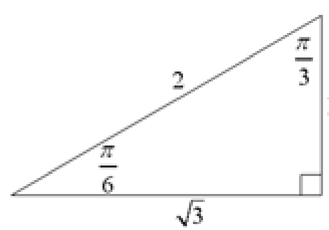
5.2

Special Triangles

I)



II)

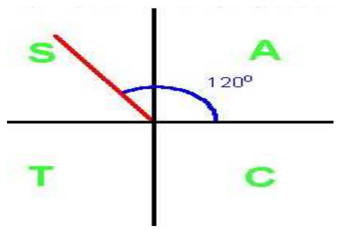


Using the "special" triangles above, we can find the exact trigonometric ratios for angles of pi/3, pi/4 and pi/6. These triangles can be constructed quite easily and provide a simple way of remembering the trigonometric ratios. The table below lists some of the more common angles (in both radians and degrees) and their exact trigonometric ratios.

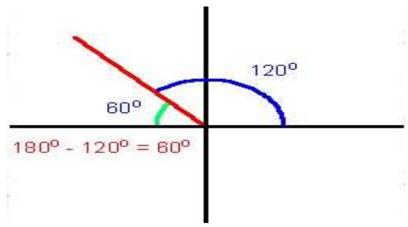
θ	*rad*	$\sin\theta$	$\cos\theta$	$\tan\theta$
0°	0	0	1	0
30°	$\pi/6$	$1/2$	$\sqrt{3}/2$	$\sqrt{3}/3$
45°	$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$
90°	$\pi/2$	1	0	–

Ex. 1 Find the exact value of $\sin 120^\circ$.

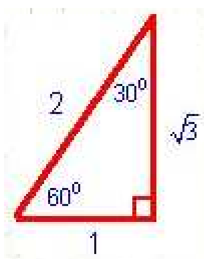
Step #1: Draw the given angle in standard position.



Step #2: Find the reference angle.



Step #3: Use the 60 degree special triangle to calculate the sin 120.



$$\sin 60^0 = \frac{\sqrt{3}}{2}$$

Step #4: Use the CAST principal to determine if the trigonometric ratio should be positive or negative. In quadarnt two sin is positive therefore we get:

$$\sin 120^0 = +\frac{\sqrt{3}}{2}$$

U.5: Periodic Functions, Trig Functions and trig identities

5.3

Periodic Functions

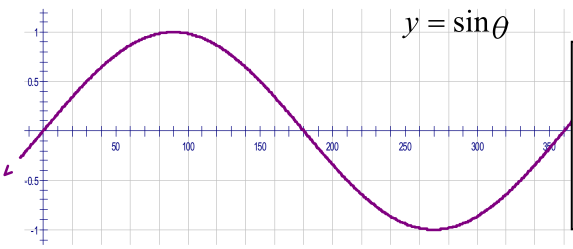
A periodic function has a repeating pattern.
The **cycle** is the smallest complete repeating pattern.
The **axis of the curve** is a horizontal line that is midway between the maximum and minimum values of the graph. The equation is $y = \dfrac{\text{max value} + \text{min value}}{2}$.
The **period** is the length of the cycle.
The **amplitude** is the magnitude of the vertical distance from the axis of the curve to the

maximum or minimum value. The equation is $a = \dfrac{\text{max value} - \text{min value}}{2}$

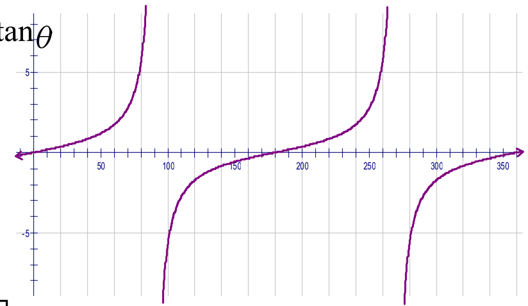
Sin Cos Tan Graphs

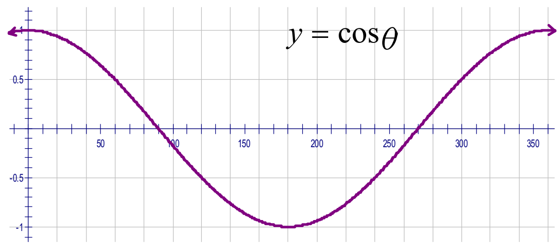
Trigonometric Functions

The graphs of $y = \sin\theta$, $y = \cos\theta$, and $y = \tan\theta$ are shown below.



$y = \sin\theta$
Period = 360°
Amplitude = 1
Zeroes = 0°, 180°, 360°…





$y = \cos\theta$
Period = 360°
Amplitude = 1
Zeroes = 90°, 270°…

$y = \tan\theta$
Period = 180°
Zeroes = 0°, 180°, 360°…
Vertical asymptotes = 90°, 270°…

5.6

Transformations of Trigonometric Functions

Transformations apply to trig functions as they do to any other function.
The graphs of $y = a\sin k(\theta + b) + d$ and $y = a\cos k(\theta + b) + d$ are transformations of the graphs $y = \sin\theta$ and $y = \cos\theta$ respectively.
The value of a determines the vertical stretch, called the **amplitude**.
It also tells whether the curve is reflected in the θ -axis.

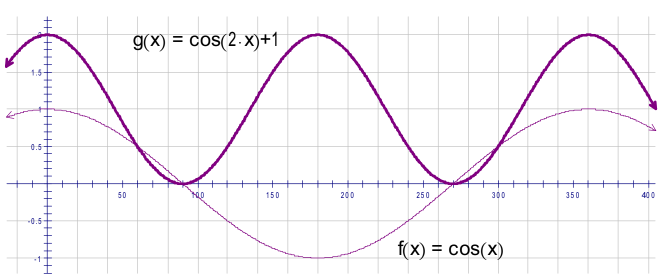
The value of k determines the horizontal stretch. The graph is stretched by a factor of $\frac{1}{k}$. We can use this value to determine the **period** of the transformation of $y = \sin\theta$ or $y = \cos\theta$.

The period of $y = \sin k\theta$ or $y = \cos k\theta$ is $\dfrac{360^\circ}{k}$, $k > 0$. The period of $y = \tan k\theta$ is $\dfrac{180^\circ}{k}$, $k > 0$.

The value of b determines the horizontal translation, known as the **phase shift**.
The value of d determines the vertical translation. $y = d$ is the equation of the **axis of the curve**.

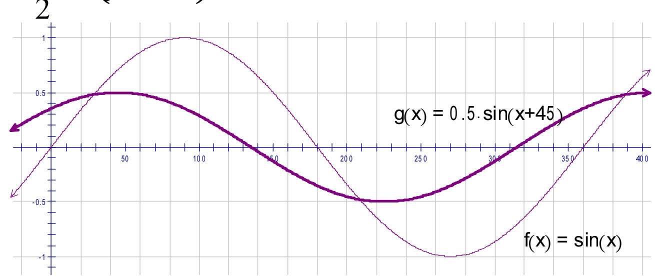
e.g.

$y = \cos 2\theta + 1$



e.g.

$y = \frac{1}{2}\sin(\theta + 45^\circ)$



5.7

Trigonometric Identities

Pythagorean Identity: $\sin^2\theta + \cos^2\theta = 1$ OR $\sin^2\theta = 1 - \cos^2\theta$

Quotient Identity: $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ OR $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$

Also $\csc\theta = 1/\sin\theta$ $\sec\theta = 1/\cos\theta$ $\cot\theta = 1/\tan\theta$

e.g. Prove the identity. $\sin^2\theta + 2\cos^2\theta - 1 = \cos^2\theta$

Work with each side separately. Look for the quotient or Pythagorean identities. You may need to factor, simplify or split terms up. When you are done, write a concluding statement.

$LS = \sin^2\theta + 2\cos^2\theta - 1$

$= \sin^2\theta + \cos^2\theta + \cos^2\theta - 1$

$= 1 + \cos^2\theta - 1$

$= \cos^2\theta = RS$ Since LS=RS then $\sin^2\theta + 2\cos^2\theta - 1 = \cos^2\theta$ is true for all values of θ.

5.8 Trig Equations

A **trigonometric equation** is one that involves one or more of the six functions sine, cosine, tangent, cotangent, secant, and cosecant. Some trigonometric equations, like x = cos x, can be solved only numerically. But a great many can be solved in closed form, and this page shows you how to do it in five steps.

Ex.

$$2\sin(x) - 1 = 0$$

$$\sin(x) = \frac{1}{2} \quad x = \frac{\pi}{6}.$$

The sine function is positive in quadrants I and II. The $\sin\left(\pi - \frac{\pi}{6}\right) = \sin\frac{5\pi}{6}$ is also equal to $\frac{1}{2}$.

Therefore, two of the solutions to the problem are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

U.6: Sequences and the Binomial Theorem

6.1

A sequence is a list of numbers that has a certain order. For example, 16,7,888,-2,0,72 is a sequence, since 16 is the first number in the list, 7 is the second, etc. Sequences can have a certain "rule" by which terms progress, but they can also be completely random.

1,4,9,16… a sequence of the perfect squares starting from 1.

t_1 = term 1 t_2 = term 2 and so on

t_n = nth term or the general term

*Sequences can also appear as functions

i.e. tn = 1-2n is the same as f(n) = 1-2n

6.2

Arithmetic sequences are sequences that start with any number a, and in which every n^{th} term can be written as tn $= a + (n-1)d$, where d is any number. An example of such a sequence would be 5, 12, 19, 26, 33…, where $a = 5$ and $d = 7$. This is an increasing arithmetic sequence, as the terms are increasing. Decreasing arithmetic sequences have $d < 0$.

6.3

Geometric sequences also start with any number a (though usually a is nonzero here), but this time we are not adding an extra d value each time- we multiply a by a factor of r. Thus, the n^{th} term is tn $= ar^{n-1}$. Geometric sequences can either be monotonic, when r is positive and the terms are moving in one direction, or alternating, where $r < 0$ and the terms alternate between positive and negative values, depending on n.

6.4

Recursion is the process of choosing a starting term and repeatedly applying the same process to each term to arrive at the following term. Recursion requires that you know the value of the term immediately before the term you are trying to find.

A recursive formula always has two parts:

1. the starting value for a_1.
2. the recursion equation for a_n as a function of a_{n-1} (the term before it.)

Recursive formula:	Same recursive formula:
$a_1 = 4$ $a_n = 2a_{n-1}$	$a_1 = 4$ $a_{n+1} = 2a_n$

6.5

A series is a sequence of numbers that represent partial sums for another sequence. For example, if my sequence is 1,2,3,4... then my series would be 1,1+2,1+2+3,..., or 1,3,6,10....

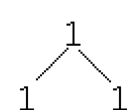
Arithmetic Series Formula: $s_n = \frac{n(2a+(n-1)d)}{2}$

6.6

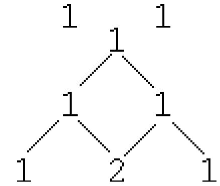
Geometric Series Formula: $s_n \frac{a(1-r^n)}{1-r}$

Binomial Theorem

Pascal's Triangle". To make the triangle, you start with a pyramid of three 1's, like this:

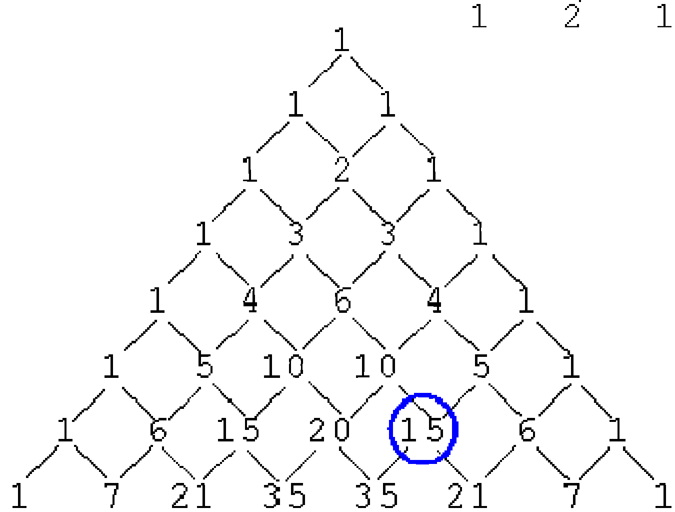


Then you get the next row of numbers by adding the pairs of numbers from above. (Where there is only one number above, you just carry down the 1.)



Keep going, always adding pairs of numbers from the previous row..

the power of the binomial is the row that you want to look at. The numbers in the row will be your coefficients



The powers on each term in the expansion always added up to whatever n was, and that the terms counted up from zero to n.

Ex. Expand $(x^2 + 3)^6$

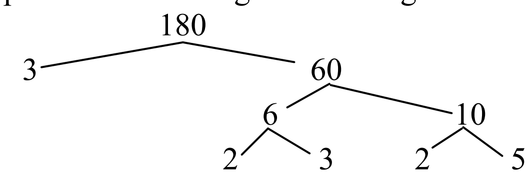
$(\mathbf{1})(x^{12})(1) + (\mathbf{6})(x^{10})(3) + (\mathbf{15})(x^8)(9) + (\mathbf{20})(x^6)(27)$ *Coefficients are from P's Triangle

$+ (\mathbf{15})(x^4)(81) + (\mathbf{6})(x^2)(243) + (\mathbf{1})(1)(729)$

$= \mathbf{x^{12} + 18x^{10} + 135x^8 + 540x^6 + 1215x^4 + 1458x^2 + 729}$

Prime Factorization Factor a number into its prime factors using the tree diagram method.

e.g.



$180 = (2^2)(3^2)(5)$

e.g. Evaluate.

$$(3^0 + 3^2)^{-2} = (1+9)^{-2} = 10^{-2} = \frac{1}{10^2} = \frac{1}{100}$$

Follow the order of operations. Evaluate brackets first.

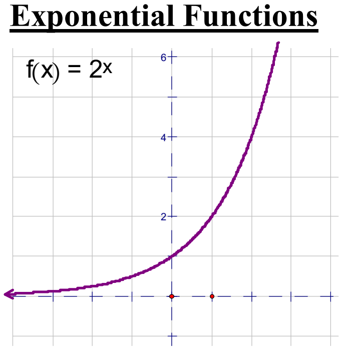
e.g. Simplify.

$$\left(\frac{b^3}{2a^{-3}}\right)^{-2} = \frac{b^{3(-2)}}{(2a^{-3})^{-2}} = \frac{b^{-6}}{2^{-2}a^{-3(-2)}} = \frac{2^2 b^{-6}}{a^6} = \frac{4}{a^6 b^6}$$

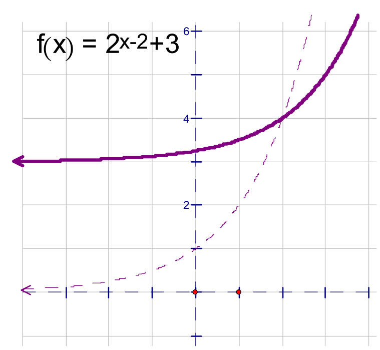
Power of a quotient.

Power of a product.

Exponential Functions



In general, the exponential function is defined by the equation, $y = a^x$ or $f(x) = a^x$, $a > 0, x \in R$. Transformations apply to exponential functions the same way they do to all other functions.



Compound Interest

Calculating the future amount: $A = P(1+i)^n$

Calculating the present amount: $P = A(1+i)^{-n}$

A – future amount

P – present (initial) amount

i – interest rate per conversion period

n – number of conversion periods