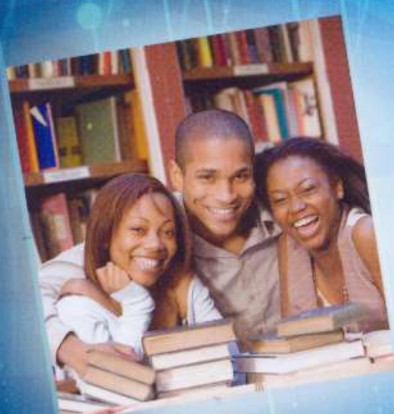
Math 12
Calculus and Vectors
Ontario Edition

THE KEY STUDY GUIDE



- 100% aligned with the Ontario curriculum
- Includes practice questions and tests
- Contains answers and detailed solutions
- Complements classroom instruction
- Reviewed by respected Ontario educators

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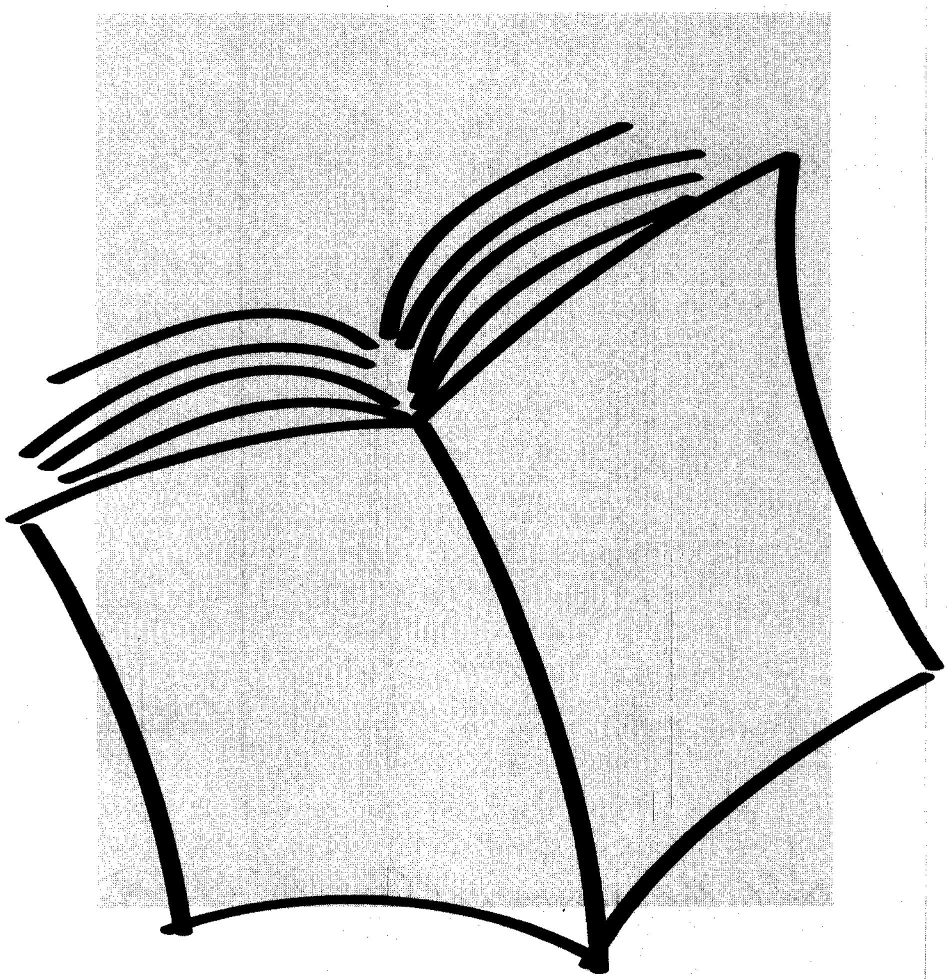
THE KEY

STUDENT STUDY GUIDE

THE KEY student study guide is designed to help students achieve success in school. The content in each study guide is 100% curriculum aligned and serves as an excellent source of material for review and practice. To create this book, teachers, curriculum specialists, and assessment experts have worked closely to develop the instructional pieces that explain each of the key concepts for the course. The practice questions and sample tests have detailed solutions that show problem-solving methods, highlight concepts that are likely to be tested, and point out potential sources of errors. ***THE KEY*** is a complete guide to be used by students throughout the school year for reviewing and understanding course content, and to prepare for assessments.



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THE KEY – Math 12 - Calculus and Vectors Ontario

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Dedicated to the memory of Dr. V. S. Rao

THE KEY–Math 12 Calculus and Vectors

THE KEY consists of the following sections:

KEY Tips for Being Successful at School gives examples of study and review strategies. It includes information about learning styles, study schedules, and note taking for test preparation.

Class Focus includes a unit on each area of the curriculum. Units are divided into sections, each focusing on one of the specific expectations, or main ideas, that students must learn about in that unit. Examples, definitions, and visuals help to explain each main idea. Practice questions on the main ideas are also included. At the end of each unit is a test on the important ideas covered. The practice questions and unit tests help students identify areas they know and those they need to study more. They can also be used as preparation for tests and quizzes. Most questions are of average difficulty, though some are easy and some are hard—the harder questions are called *Challenger Questions*. Each unit is prefaced by a ***Table of Correlations***, which correlates questions in the unit to the specific curriculum expectations. Answers and solutions are found at the end of each unit.

KEY Strategies for Success on Tests helps students get ready for tests. It shows students different types of questions they might see, word clues to look for when reading them, and hints for answering them.

Practice Tests includes one to three tests based on the entire course. They are very similar to the format and level of difficulty that students may encounter on final tests. In some regions, these tests may be reprinted versions of official tests, or reflect the same difficulty levels and formats as official versions. This gives students the chance to practice using real-world examples. Answers and complete solutions are provided at the end of the section.

For the complete curriculum document (including specific expectations along with examples and sample problems), visit www.edu.gov.on.ca/eng/curriculum/secondary.

THE KEY *Study Guides* are available for many courses. Check www.castlerockresearch.com for a complete listing of books available for your area.

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At Castle Rock Research, we strive to produce an error-free resource. If you should find an error, please contact us so that future editions can be corrected.

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KEY TIPS FOR BEING SUCCESSFUL AT SCHOOL

KEY FACTORS CONTRIBUTING TO SCHOOL SUCCESS

In addition to learning the content of your courses, there are some other things that you can do to help you do your best at school. Some of these strategies are listed below.

- **Keep a positive attitude:** Always reflect on what you can already do and what you already know.
- **Be prepared to learn**: Have ready the necessary pencils, pens, notebooks, and other required materials for participating in class.
- **Complete all of your assignments:** Do your best to finish all of your assignments. Even if you know the material well, practice will reinforce your knowledge. If an assignment or question is difficult for you, work through it as far as you can so that your teacher can see exactly where you are having difficulty.
- **Set small goals for yourself when you are learning new material:** For example, when learning the parts of speech, do not try to learn everything in one night. Work on only one part or section each study session. When you have memorized one particular part of speech and understand it, then move on to another one, continue this process until you have memorized and learned all the parts of speech.
- **Review your classroom work regularly at home:** Review to be sure that you understand the material that you learned in class.
- **Ask your teacher for help**: Your teacher will help you if you do not understand something or if you are having a difficult time completing your assignments.
- **Get plenty of rest and exercise:** Concentrating in class is hard work. It is important to be well-rested and have time to relax and socialize with your friends. This helps you to keep your positive attitude about your school work.
- **Eat healthy meals:** A balanced diet keeps you healthy and gives you the energy that you need for studying at school and at home.

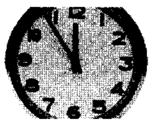
How To Find Your Learning Style

Every student learns differently. The manner in which you learn best is called your learning style. By knowing your learning style, you can increase your success at school. Most students use a combination of learning styles. Do you know what type of learner you are? Read the following descriptions. Which of these common learning styles do you use most often?

- **Linguistic Learner**: You may learn best by saying, hearing, and seeing words. You are probably really good at memorizing things such as dates, places, names, and facts. You may need **to write and then say out loud** the steps in a process, a formula, or the actions that lead up to a significant event.

- **Spatial Learner**: You may learn best by looking at and working with pictures. You are probably really good at puzzles, imagining things, and reading maps and charts. You may need to use strategies like **mind mapping and webbing** to organize your information and study notes.

- **Kinaesthetic Learner**: You may learn best by touching, moving, and figuring things out using manipulative. You are probably really good at physical activities and learning through movement. You may need to **draw your finger over a diagram** to remember it, **"tap out" the steps** needed to solve a problem, or **"feel" yourself writing** or typing a formula.



Scheduling Study Time

You should review your class notes regularly to ensure that you have a clear understanding of all the new material you learned. Reviewing your lessons on a regular basis helps you to learn and remember ideas and concepts. It also reduces the quantity of material that you need to study prior to a test. Establishing a study schedule will help you to make the best use of your time.

Regardless of the type of study schedule you use, you may want to consider the following suggestions to maximize your study time and effort:

- Organize your work so that you begin with the most challenging material first.
- Divide the subject's content into small, manageable chunks.
- Alternate regularly between your different subjects and types of study activities in order to maintain your interest and motivation.
- Make a daily list with headings like "Must Do," "Should Do," and "Could Do."
- Begin each study session by quickly reviewing what you studied the day before.
- Maintain your usual routine of eating, sleeping, and exercising to help you concentrate better for extended periods of time.

Creating Study Notes

Mind-Mapping or Webbing

Use the key words, ideas, or concepts from your reading or class notes to create a *mind map* or *web* (a diagram or visual representation of the given information). A mind map or web is sometimes referred to as a knowledge map.

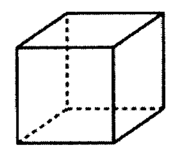
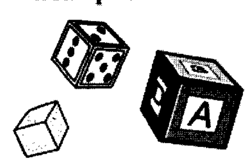
- Write the key word, concept, theory, or formula in the centre of your page.
- Write down related facts, ideas, events, and information and then link them to the central concept with lines.
- Use coloured markers, underlining, or other symbols to emphasize things such as relationships, time lines, and important information.

The following examples of a Frayer Model illustrate how this technique can be used to study mathematical vocabulary.

Perimeter

Definition – Perimeter is the distance around a polygon	**Characteristics** – Measured in linear units (e.g., metres, centimetres)
Examples – Fence around a yard – Distance around a circle (circumference)	**Non-examples** – Grass covering a yard – Area of rug covering a floor

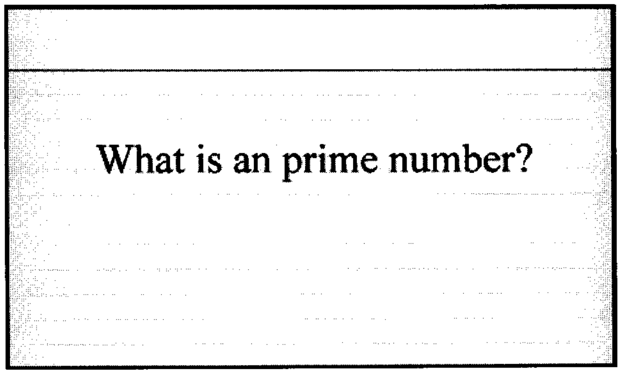
Cube

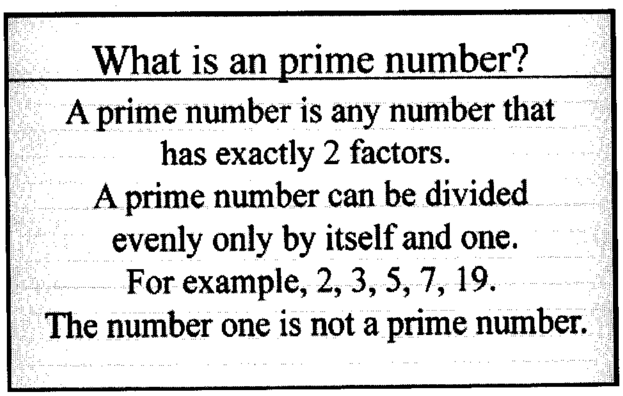
Definition A cube is a solid 3-D object that has – 6 square faces, all equal in size – 8 vertices – 12 equal edges	**Visual Presentation** 
Characteristics or Properties – 6 square faces – 8 vertices – 12 edges – 6 flat faces	**Examples** 

Index Cards

To use index cards while studying, follow these steps:

- Write a key word or question on one side of an index card.
- On the reverse side, write the definition of the word, answer to the question, or any other important information that you want to remember.





Symbols and Sticky Notes—Identifying Important Information

- Use symbols to mark your class notes. For example, an exclamation mark (!) might be used to point out something that must be learned well because it is a very important idea. A question mark (?) may highlight something that you are not certain about, and a diamond (◊) or asterisk (*) could highlight interesting information that you want to remember.
- Use sticky notes when you are not allowed to put marks in books.
- Use sticky notes to mark a page in a book that contains an important diagram, formula, explanation, etc.
- Use sticky notes to mark important facts in research books.

Memorization Techniques

- **Association** relates new learning to something you already know. For example, to remember the spelling difference between *dessert* and *desert*, recall that the word *sand* has only one *s*. So, because there is sand in a desert, the word *desert* only has on *s*.
- **Mnemonic** devices are sentences that you create to remember a list or group of items. For example, the first letter of each word in the phrase "**E**very **G**ood **B**oy **D**eserves **F**udge" helps you to remember the names of the lines on the treble clef staff (E, G, B, D, and F) in music.
- **Acronyms** are words that are formed from the first letters or parts of the words in a group. For example, **RADAR** is actually an acronym for **Ra**dio **D**etecting **A**nd **R**anging, and **MASH** is an acronym for **M**obile **A**rmy **S**urgical **H**ospital. **HOMES** helps you to remember the names of the five Great Lakes (**H**uron, **O**ntario, **M**ichigan, **E**rie, and **S**uperior).
- **Visualizing** requires you to use your mind's eye to "see" a chart, list, map, diagram, or sentence as it is in your textbook or notes, on the chalk board or computer screen, or in a display.
- **Initialisms** are abbreviations that are formed from the first letters or parts of the words in a group. Unlike acronyms, initialisms cannot be pronounced as a word themselves. For example, **BEDMAS** is an initialism for the order of operations in math (**B**rackets, **E**xponents, **D**ivide, **M**ultiply, **A**dd, **S**ubtract).

Key Strategies for Reviewing

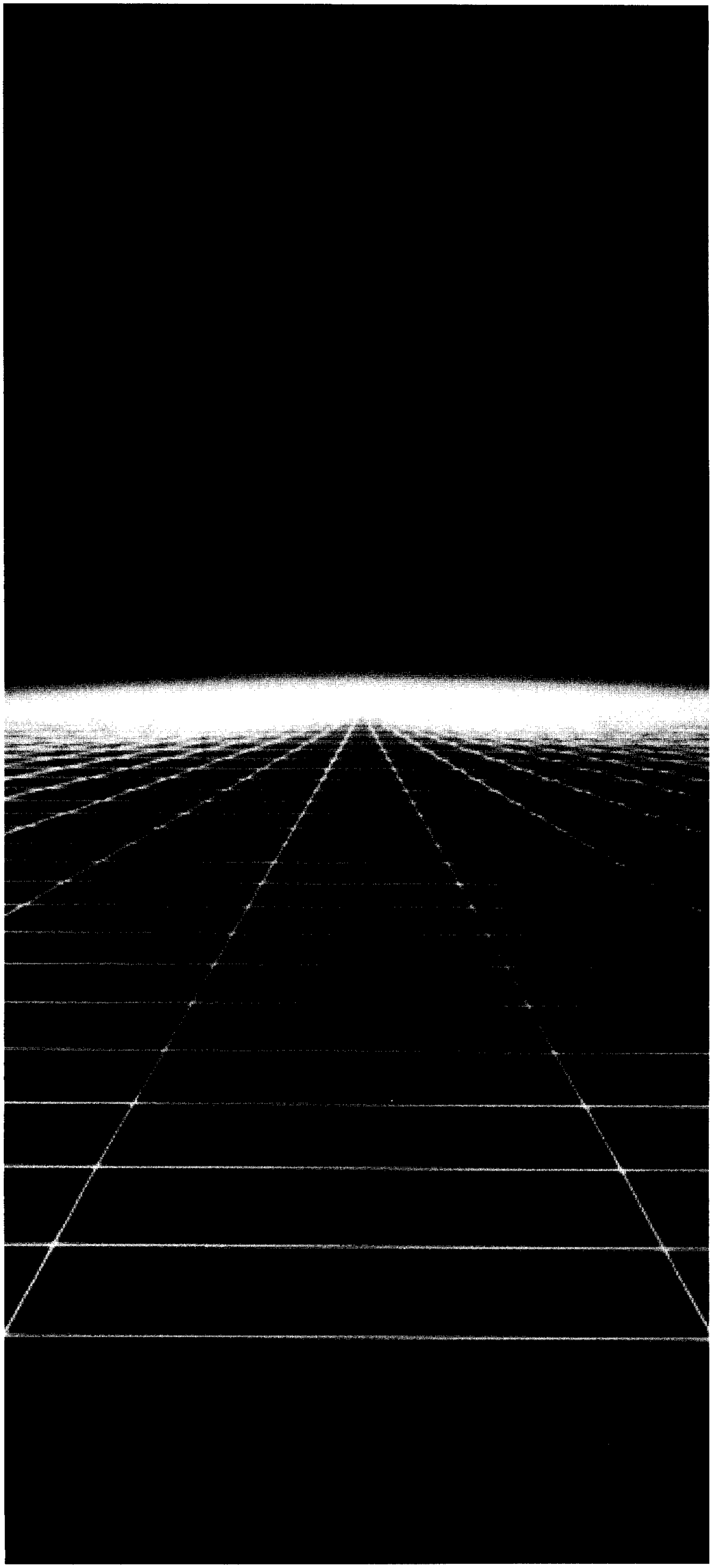
Reviewing textbook material, class notes, and handouts should be an ongoing activity. Spending time reviewing becomes more critical when you are preparing for tests. You may find some of the following review strategies useful when studying during your scheduled study time.

- Before reading a selection, preview it by noting the headings, charts, graphs, and chapter questions.
- Read the complete introduction to identify the key information that is addressed in the selection.
- Read the first sentence of the next paragraph for the main idea.
- Skim the paragraph and note the key words, phrases, and information.
- Read the last sentence of the paragraph.
- Repeat this process for each paragraph and section until you have skimmed the entire selection.

KEY Strategies for Success: A Checklist

Review, review, review: review is a huge part of doing well at school and preparing for tests. Here is a checklist for you to keep track of how many suggested strategies for success you are using. Read each question and then put a check mark (✓) in the correct column. Look at the questions where you have checked the "No" column. Think about how you might try using some of these strategies to help you do your best at school.

***KEY* Strategies for Success**	**Yes**	**No**
Do you attend school regularly?		
Do you know your personal learning style—how you learn best?		
Do you spend 15 to 30 minutes a day reviewing your notes?		
Do you study in a quiet place at home?		
Do you clearly mark the most important ideas in your study notes?		
Do you use sticky notes to mark texts and research books?		
Do you practise answering multiple-choice and written-response questions?		
Do you ask your teacher for help when you need it?		
Are you maintaining a healthy diet and sleep routine?		
Are you participating in regular physical activity?		



Rate of Change

Rate of Change

Table of Correlations

	Outcome	Practice Questions	Unit Test Questions	Practice Test 1	Practice Test 2
RC1.0	Investigating Instantaneous Rate of Change at a Point				
RC1.1	*describe examples of real-world applications of rates of change, represented in a variety of ways*	1, 2	1, 2		1
RC1.2	*describe connections between the average rate of change of a function that is smooth (i.e., continuous with no corners) over an interval and the slope of the corresponding secant, and between the instantaneous rate of change of a smooth function at a point and the slope of the tangent at that point*	3, 4	3, 4	1	2
RC1.3	*make connections, with or without graphing technology, between an approximate value of the instantaneous rate of change at a given point on the graph of a smooth function and average rates of change over intervals containing the point (i.e., by using secants through the given point on a smooth curve to approach the tangent at that point, and determining the slopes of the approaching secants to approximate the slope of the tangent)*	5, 6	5, 6	2	
RC1.4	*recognize, through investigation with or without technology, graphical and numerical examples of limits, and explain the reasoning involved*	7	7		3
RC1.5	*make connections, for a function that is smooth over the interval a [lesser than or equal to symbol] x [lesser than or equal to symbol] a + h, between the average rate of change of the function over this interval and the value of the expression f(a + h) – f(a)/h , and between the instantaneous rate of change of the function at x = a and the value of the limit lim[h arrow 0] f(a + h) – f(a)/h*	8, 9	8, 9	3	4
RC1.6	*compare, through investigation, the calculation of instantaneous rates of change at a point (a, f(a)) for polynomial functions, with and without f(a + h) – f(a)/h simplifying the expression before substituting values of that approach zero*	10, 11	10, 11	4	5
RC2.0	Investigating the Concept of the Derivative Function				
RC2.1	*determine numerically and graphically the intervals over which the instantaneous rate of change is positive, negative, or zero for a function that is smooth over these intervals (e.g., by using graphing technology to examine the table of values and the slopes of tangents for a function whose equation is given; by examining a given graph), and describe the behaviour of the instantaneous rate of change at and between local maxima and minima*	12, 13, 14	12, 13, 14	5, 6	6
RC2.2	*generate, through investigation using technology, a table of values showing the instantaneous rate of change of a polynomial function, f(x), for various values of x (e.g., construct a tangent to the function, measure its slope, and create a slider or animation to move the point of tangency), graph the ordered pairs, recognize that the graph represents a function called the derivative, f′(x) or dy/dx, and make connections between the graphs of f(x) and f′(x) or y and dy/dx [e.g., when f(x) is linear, f′(x) is constant; when f(x) is quadratic, f′(x) is linear; when f(x) is cubic, f′(x) is quadratic]*	15	15		

RC2.3	*determine the derivatives of polynomial functions by simplifying the algebraic expression f(x + h) – f(x)/h and then taking the limit of the simplified expression as h approaches zero [i.e., determining lim[h arrow 0] f(x + h) – f(x)/h]*	16, 17	16, 17	7	7
RC2.4	*determine, through investigation using technology, the graph of the derivative f'(x)or dy of a given sinusoidal function [i.e., dx f(x) = sin x, f(x) = cos x] (e.g., by generating a table of values showing the instantaneous rate of change of the function for various values of x and graphing the ordered pairs; by using dynamic geometry software to verify graphically that when f(x) = sin x, f'(x) = cos x, and when f(x) = cos x, f'(x) = –sin x; by using a motion sensor to compare the displacement and velocity of a pendulum)*	18	18, 19		
RC2.5	*determine, through investigation using technology, the graph of the derivative f'(x)or dy/dx of a given exponential function [i.e., f(x) = a(x) (a > 0, a ≠ 1)], and make connections between the graphs of f(x) and f'(x) or y and dy/dx*	19, 20	20, 21		8
RC2.6	*determine, through investigation using technology, the exponential function f(x) = a(x) (a > 0, a ≠ 1) for which f'(x) = f(x) (e.g., by using graphing technology to create a slider that varies the value of a in order to determine the exponential function whose graph is the same as the graph of its derivative), identify the number e to be the value of a for which f'(x) = f(x) [i.e., given f(x) = e(x), f (x) = e(x)], and recognize that for the exponential function f(x) = e(x) the slope of the tangent at any point on the function is equal to the value of the function at that point*	21	22	8	9
RC2.7	*recognize that the natural logarithmic function f(x) = log(e)x, also written as f(x) = ln x, is the inverse of the exponential function f(x) = e(x), and make connections between f(x) = ln x and f(x) = e(x)*	22	23	9	10
RC2.8	*verify, using technology (e.g., calculator, graphing technology), that the derivative of the exponential function f(x) = a(x) is f'(x) = ax ln a for various values of a*	23	24	10, 11	11
RC3.0	**Investigating the Properties of Derivatives**				
RC3.1	*verify the power rule for functions of the form f(x) = x(n), where n is a natural number*	24	25	12	12
RC3.2	*verify the constant, constant multiple, sum, and difference rules graphically and numerically, and read and interpret proofs involving lim[h arrow 0] f(x + h) – f(x)/h of the constant, constant multiple, sum, and difference rules (student reproduction of the development of the general case is not required)*	25, 26	26, 36	13	13
RC3.3	*determine algebraically the derivatives of polynomial functions, and use these derivatives to determine the instantaneous rate of change at a point and to determine point(s) at which a given rate of change occurs*	27, 28, 29, 30	27, 28, 29, 37	14	14
RC3.4	*verify that the power rule applies to functions of the form f(x) = x(n), where n is a rational number, and verify algebraically the chain rule using monomial functions and the product rule using polynomial functions*	31	30	15	15
RC3.5	*solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions, radical functions, and other simple combinations of functions**	32, 33, 34, 35, 36	31, 32, 33, 34, 35	16	16

RC1.1 describe examples of real-world applications of rates of change, represented in a variety of ways

APPLICATIONS OF RATES OF CHANGE

There are numerous applications and situations involving rate of change.

With the present-day interest in global warming, there are many historical studies and a lot of research being done on the rate at which various regions of Earth are warming. On a related topic, there is research on how fast the ozone layer of the atmosphere is being depleted.

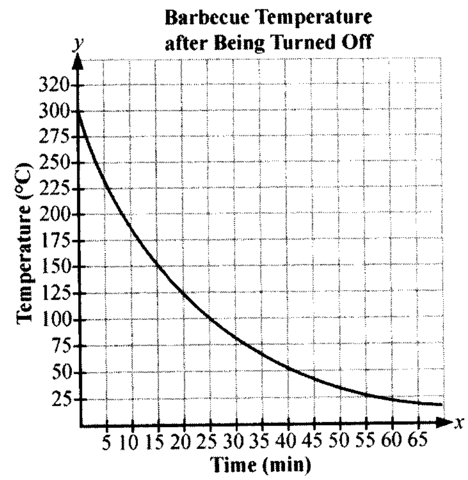
In medicine, there are studies on the rate of advancement of disease, and the rate of effectiveness of drugs and other treatments.

In finance and economics, there is interest in many rates of change, such as the rate of growth of economies and the rate of change of the values of stock markets.

The rate of change of displacement with respect to time is called velocity. When an object is dropped from a height of 200 m, its velocity, v, in metres per second, toward the centre of Earth after t seconds is given by the formula $v = -9.8t$. This formula disregards the effect of air friction on the object. Thus, after 3 s, the velocity of the object is $v = -9.8(3) = -29.4$. The negative sign indicates the direction is toward Earth rather than away from it.

Graphs are often used to show or emphasize rates of change. For example, consider what happens to the temperature in a barbecue when it is turned off.

This graph shows the temperature, y (measured in degrees Celsius), in a particular barbecue x minutes after it has been turned off.



The temperature of the barbecue falls quickly initially (the graph is steep) and much more slowly as time passes (the graph is getting flatter). This shows that the rate of change of the temperature with respect to time is greater initially (negatively) than it is after it gets closer to the temperature of the surrounding air.

1. Which of the following changes does **not** represent a positive rate of change?

A. The change in temperature of water being heated over time

B. The change in fuel economy as a vehicle increases its speed

C. The change in tension of a spring as the spring is compressed

D. The change in average monthly temperature in Canada from January to May

Use the following information to answer the next question.

An object's position, s, in metres with respect to time, t, in seconds is given by the formula $s = 2t - 5$.

2. What is the rate of change of the object from 0 to 5 s?

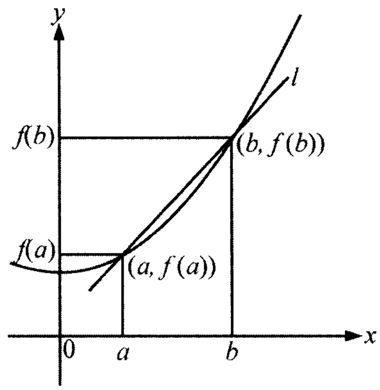
A. 0.0 m/s **B.** 2.0 m/s

C. 2.5 m/s **D.** 5.0 m/s

RC1.2 describe connections between the average rate of change of a function that is smooth (i.e., continuous with no corners) over an interval and the slope of the corresponding secant, and between the instantaneous rate of change of a smooth function at a point and the slope of the tangent at that point

Understanding Average Rate of Change and Instantaneous Rate of Change

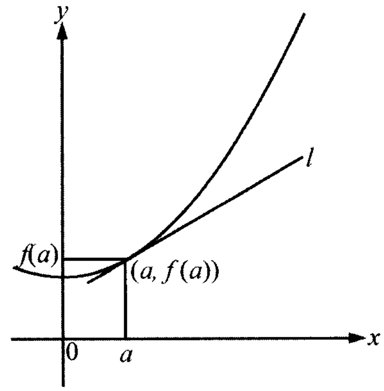
The **average rate of change** of a continuous function $f(x)$ over an interval from $x = a$ to $x = b$ is determined by calculating $\frac{f(b) - f(a)}{b - a}$. This is also the slope, m_S, of the secant line, l, through points $(a, f(a))$ and $(b, f(b))$.



Therefore, the average rate of change over an interval from $x = a$ to $x = b$ is $m_S = \frac{f(b) - f(a)}{b - a}$.

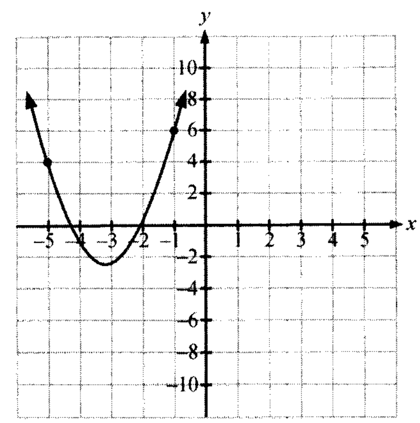
The slope of the secant line gets closer to the slope of the tangent line at point $(a, f(a))$ as point $(b, f(b))$ moves closer to point $(a, f(a))$. Therefore, the **instantaneous rate of change** of the function at the point $(a, f(a))$ is the slope of the tangent line, m_T, at $(a, f(a))$. This can also be described as the instantaneous rate of change of y (or $f(x)$) with respect to x when $x = a$.

This diagram shows the tangent line at $(a, f(a))$.



Use the following information to answer the next question.

The graph of a function, f, is given.



3. What is the average rate of change of f between the points $(-1, f(-1))$ and $(-5, f(-5))$?

   A. -2

   B. $-\frac{1}{2}$

   C. $\frac{1}{2}$

   D. 2

Written Response

4. For a smooth function $y = f(x)$, what does the quotient $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ represent?

RC1.3 make connections, with or without graphing technology, between an approximate value of the instantaneous rate of change at a given point on the graph of a smooth function and average rates of change over intervals containing the point (i.e., by using secants through the given point on a smooth curve to approach the tangent at that point, and determining the slopes of the approaching secants to approximate the slope of the tangent)

Approximating Instantaneous Rates of Change

The instantaneous rate of change or slope of a tangent at a fixed point of a graph of a function can be approximated by calculating the slopes of secants as one point approaches the fixed point.

The TABLE feature on a TI-83 Plus graphing calculator can be used to find the instantaneous rate of change of a given function at a particular x-value.

Example

Using the TABLE feature of a TI-83 Plus calculator and x-values of -1.1, -0.9, -1.01, -0.99, -1.001, and -0.999, estimate the instantaneous rate of change of y with respect to x for $y = x^2 - 3x$ at $x = -1$ by finding the slopes of secant lines of the corresponding graph.

Solution

Before solving the problem, set up your calculator by clearing the table entries. Press [2nd] [0], scroll down to ClrTable, and press [ENTER] twice.

Step 1
Determine the y-coordinate when $x = -1$.
$y = (-1)^2 - 3(-1)$
$y = 4$
Therefore, the stationary point for the secant lines is $(-1, 4)$.

Step 2
Enter the equations into the calculator.
Press [Y=], and enter $Y_1 = X\text{^}2 - 3X$.
To enter $Y_2 = (Y_1 - 4)/(X - (-1))$, place the cursor into $Y_2 =$. Press [(] and then [VARS]. Scroll right to access the Y-Vars Menu. Select 1:Function…, and press [ENTER] twice to bring up Y_1. Then, enter the rest of the equation.

Step 3
Generate a table for analysis.
Press [2nd] [WINDOW]. Highlight Indpnt: ASK, Depend: AUTO.
Press [2nd] [GRAPH].
Type the six x-values, -1.1, -0.9, -1.01, -0.99, -1.001, and -0.999 into the first column, pressing [ENTER] after each one. You should see the window shown here.

X	Y1	Y2
-1.1	4.51	-5.1
-.9	3.51	-4.9
-1.01	4.0501	-5.01
-.99	3.9501	-4.99
-1.001	4.005	-5.001
-.999	3.995	-4.999

X=

Step 4
Analyze the table.
The second function will give the slopes of the secant lines for the various x-values of the second point.
From the slopes of the secant lines (Y_2 in the table), you can see that the slope of the tangent line at $x = -1$ is approximately -5. Therefore, the instantaneous rate of change of y with respect to x at $x = -1$ can be estimated to be -5.

Use the following information to answer the next question.

The instantaneous rate of change of a function at $x = -3$ is 5.5.

5. Which of the following pairs of points on the graph of the function could be used to **best** approximate the instantaneous rate of change of the function at $x = -3$?

A. $(-4, -16)$ and $(-2, -4)$

B. $(-3.5, -11.31)$ and $(-3, -8)$

C. $(-3, -8)$ and $(-2.5, -5.657)$

D. $(-3.5, -11.31)$ and $(-2.5, -5.657)$

Numerical Response

6. The instantaneous rate of change of the function $y = 2x^3 - 5$ at $x = 2$, to the nearest whole number, is _________.

RC1.4 recognize, through investigation with or without technology, graphical and numerical examples of limits, and explain the reasoning involved

UNDERSTANDING LIMITS

A **limit** is a boundary that can be approached, possibly met, and usually not exceeded.
This boundary may be a number or a line or curve that can be defined by a relation. Some functions approach asymptotes when the independent variable approaches positive or negative infinity or a value for which the function is not defined. Functions can also approach a numerical value rather than an asymptote when a variable approaches an undefined value.

For example, consider the function $y = \dfrac{3x^2+2}{x^2-1}$.

For very large values of $|x|$, the function approaches 3. This can be determined by changing the form of the function by dividing the numerator and denominator by x^2. The result is $y = \dfrac{3+\frac{2}{x^2}}{1-\frac{1}{x^2}}$.

In this form, you can see that as $x \to \pm\infty$, $\dfrac{2}{x^2}$ and $\dfrac{1}{x^2}$ approach 0. Therefore, $\dfrac{3+\frac{2}{x^2}}{1-\frac{1}{x^2}}$ approaches $\dfrac{3}{1} = 3$.

This means that

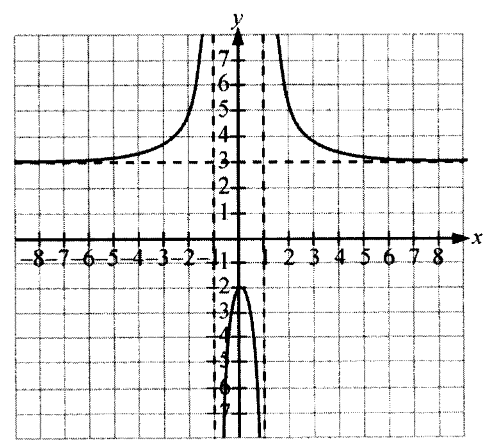
$$\lim_{x\to\infty}\frac{3x^2+2}{x^2-1} = 3 \text{ and } \lim_{x\to-\infty}\frac{3x^2+2}{x^2-1} = 3$$

Also, because the denominator of $\dfrac{3x^2+2}{x^2-1}$ cannot equal 0, $x \neq \pm 1$. As $x \to \pm 1$, the expression $\dfrac{3x^2+2}{x^2-1}$ approaches 5 divided by a number getting closer to 0, giving a result of $+\infty$.

Therefore,

$$\lim_{x\to 1}\frac{3x^2+2}{x^2-1} = \infty \text{ and } \lim_{x\to -1}\frac{3x^2+2}{x^2-1} = \infty$$

These limit results are consistent with the details of the graph of $y = \dfrac{3x^2+2}{x^2-1} = \infty$ and the graphs of its asymptotes $y = 3$, $x = 1$, and $x = -1$. These graphs, drawn using a computer graphing program, are shown here.



Use the following information to answer the next question.

The given table of values shows the values of a function over the interval $-5 \le x \le 5$, $x \in I$.

x	$f(x)$
–5	–32
–4	–16
–3	–8
–2	–4
–1	–2
0	–1
1	–0.5
2	–0.25
3	–0.125
4	–0.0625
5	–0.031 25

7. The data in the table suggests that the function has

A. at least one x-intercept

B. a hole in the graph at $x = 0$

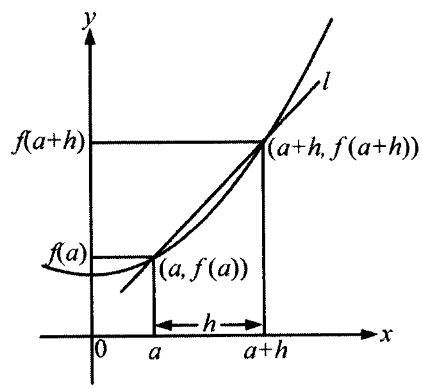
C. a vertical asymptote at $x = 0$

D. a horizontal asymptote at $y = 0$

RC1.5 make connections, for a function that is smooth over the interval a [lesser than or equal to symbol] x [lesser than or equal to symbol] a + h, between the average rate of change of the function over this interval and the value of the expression f(a + h) – f(a)/h, and between the instantaneous rate of change of the function at x = a and the value of the limit lim[h arrow 0] f(a + h) – f(a)/h

Rates of Change and Limits

The average rate of change, or slope of the secant line, of the graph of a smooth function, $f(x)$, over the interval of $(a, f(a))$ to $(b, f(b))$ is $\frac{f(b)-f(a)}{b-a}$. If $b - a$ is designated as h, $b = a + h$, and the slope of the secant line can be written as $\frac{f(a+h)-f(a)}{h}$. A graphical representation using this notation is given.



As $h \to 0$, the point $(a+h, f(a+h))$ approaches the point $(a, f(a))$, and the secant line l approaches a tangent line at $(a, f(a))$. The slope of this tangent line at $(a, f(a))$ is the instantaneous rate of change of the function at $x = a$.

Thus, the instantaneous rate of change of the function $y = f(x)$ at $x = a$ is $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$.

Written Response

8. Discuss the difference between the meaning of $\frac{f(a+h)-f(a)}{h}$ and $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ for a smooth, continuous function, *f*. Provide at least one example for which the use of each expression would be most appropriate.

Use the following information to answer the next question.

The expression $\lim_{h \to 0} \frac{\frac{2}{(-4+h)^2} - \frac{2}{(-4)^2}}{h}$ can be used to determine the instantaneous rate of change for a particular function.

Written Response

9. Determine the function $f(x)$ at the point where $x = -4$.

RC1.6 compare, through investigation, the calculation of instantaneous rates of change at a point (a, f(a)) for polynomial functions, with and without f(a + h) – f(a)/h simplifying the expression before substituting values of that approach zero

Calculating the Instantaneous Rate of Change

The instantaneous rate of change of a function, $f(x)$, at a point, $(a, f(a))$, can be predicted by using values of h that approach 0, and the expression $\frac{f(a+h)-f(a)}{h}$.

There are two methods using the expression, $\frac{f(a+h)-f(a)}{h}$, where the instantaneous rate of change can be predicted:

1. Substitute values of h that approach 0 (0.01, 0.001, 0.0001…etc) into the expression and then evaluate the expression.
2. Simplify the expression, then substitute the values of h that approach 0 (0.01, 0.001, 0.0001…etc) into the new expression.

As the value of h gets closer and closer to 0, both the simplified and non-simplified forms of the expression $\frac{f(a+h)-f(a)}{h}$ will approach a value which will be the instantaneous rate of change of a function, $f(x)$, at a point, $(a, f(a))$.

10. For which of the following functions is the instantaneous rate of change at $x = -7$ the **greatest**?

A. $f(x) = x^3$

B. $f(x) = x^3 - x$

C. $f(x) = x^2 + 9$

D. $f(x) = 2x^2 + 3x$

Written Response

11. Simplify the expression $\frac{f(a+h)-f(a)}{h}$ for the function $f(x)=x^3-1$, where $a=-5$.

RC2.1 determine numerically and graphically the intervals over which the instantaneous rate of change is positive, negative, or zero for a function that is smooth over these intervals (e.g., by using graphing technology to examine the table of values and the slopes of tangents for a function whose equation is given; by examining a given graph), and describe the behaviour of the instantaneous rate of change at and between local maxima and minima

Analyzing the Instantaneous Rate of Change with Respect to the Minima and Maxima

The instantaneous rate of change of a function can be determined from a graph or table by looking at the slopes of the tangent lines. The instantaneous rate of change of a function is:

- Positive where the slope of the tangent line is positive. This is where the graph of the function is increasing when viewed from left to right.
- Negative where the slope of the tangent line is negative. This is where the graph of the function is decreasing when viewed from left to right.
- Zero where the slope of the tangent line is zero. These points where the slope of the tangent line is zero are called *stationary points*.

The *local maxima* of a function $f(x)$ are function values where the graph achieves maximum values relative to those regions of the graph. The slopes of the tangent lines change from positive to negative at points on the graphs where the local maximums occur.

The *local minima* of a function $f(x)$ are function values where the graph achieves minimum values relative to those regions of the graph. The slopes of the tangent lines change from negative to positive at points on the graphs where the local minimums occur.

Use the following information to answer the next question.

The slope of the tangent for a smooth function is positive for $x<-3$ and $x>1$ and is negative for $-3<x<1$.

12. What is the maximum number of minima possible for the function?

A. 0 **B.** 1

C. 2 **D.** 3

Use the following information to answer the next question.

A function, f, has the following characteristics:

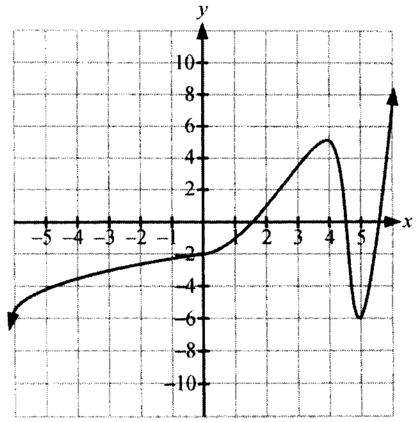
i. The instantaneous rate of change of f is 0 at $x=-5$, $x=-1$, and $x=4$.
ii. The rate of change of f is positive on $x<-5$ and $x>4$.
iii. The rate of change of f is negative on $-5<x<-1$ and $-1<x<4$.

Written Response

13. Sketch the graph of f.

Use the following information to answer the next question.

The graph of a smooth function is given.



Written Response

14. On what interval is the rate of change the **greatest** for the smooth function? Justify your answer.

RC2.2 generate, through investigation using technology, a table of values showing the instantaneous rate of change of a polynomial function, f(x), for various values of x (e.g., construct a tangent to the function, measure its slope, and create a slider or animation to move the point of tangency), graph the ordered pairs, recognize that the graph represents a function called the derivative, f′(x) or dy/dx, and make connections between the graphs of f(x) and f′(x) or y and dy/dx [e.g., when f(x) is linear, f′(x) is constant; when f(x) is quadratic, f′(x) is linear; when f(x) is cubic, f′(x) is quadratic]

Analyzing the Derivatives of Polynomial Functions Graphically

The derivative of a function for any value of its variable is the instantaneous rate of change of the function for that value of the variable. For the graph of a function, the instantaneous rate of change for any value of the variable is the slope of the tangent line at the point where that value occurs.

Given a function $f(x)$, the derivative function is commonly written in one of two ways:

1. Using prime notation—$f'(x)$ is the derivative of $f(x)$. If the function is written in the $y =$ form, the derivative can be abbreviated to y'.
2. Using delta notation—$\frac{dy}{dx}$ is the instantaneous rate of change of y with respect to the x-variable.

The derivative of a function $y = f(x)$ is

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

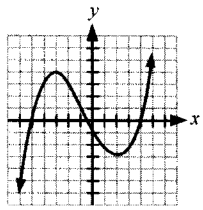
For polynomial functions of degree n, the derivative functions are of degree $n - 1$. Therefore, when $f(x)$ is linear, $f'(x)$ is constant; when $f(x)$ is quadratic, $f'(x)$ is linear; when $f(x)$ is cubic, $f'(x)$ is quadratic, and so on.

The following features can be determined when comparing the graphs of $f(x)$ and $f'(x)$:

- $f(x)$ has a local maximum or minimum when $f'(x)$ is equal to 0.
- Wherever $f'(x)$ is positive (above the x-axis), $f(x)$ increases.
- Wherever $f'(x)$ is negative (below the x-axis), $f(x)$ decreases.

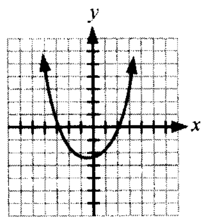
Use the following information to answer the next question.

The graph of a polynomial function $y = f(x)$ is given.

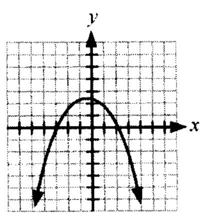


15. Which of the following graphs represents the graph of the derivative of the polynomial function $y = f(x)$?

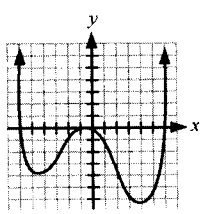
A.



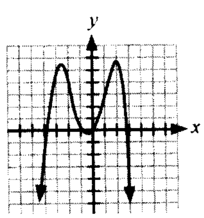
B.



C.



D.



RC2.3 determine the derivatives of polynomial functions by simplifying the algebraic expression f(x + h) – f(x)/h and then taking the limit of the simplified expression as h approaches zero [i.e., determining lim[h arrow 0] f(x + h) – f(x)/h]

Determining the Derivative of a Polynomial Function Using the Definition of the Derivative

The derivative of a specific polynomial function can be determined algebraically using the definition of the derivative, $f'(x) = \frac{dy}{dx}$ $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Use the following steps to take the derivative of a polynomial using the definition of a derivative:

1. Expand and simplify the expressions $f(x+h)$ and $f(x)$.
2. Substitute the equations for $f(x)$ and $f(x+h)$ into the equation $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, and simplify.
3. Factor h out of all the terms in the numerator.
4. Divide out the common factor, h, and determine the limit by substituting 0 for h.

16. Which of the following expressions can be used to determine the value of $f'(0)$ for $f(x) = -12x + 9$ from the definition of the derivative?

A. $\lim_{h \to 0} \frac{-12(h+9) - 9}{h}$

B. $\lim_{h \to 0} \frac{(-12h+9) - 9}{h}$

C. $\lim_{h \to 0} \frac{(-12+9)(h+0) - 9}{h}$

D. $\lim_{h \to 0} \frac{-12(h+9) - (-12+9)}{h}$

Written Response

17. When using the definition of the derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, to determine $f'(x)$ for $f(x) = x^2 + 5x - 11$, identify the simplified expression in the numerator of the limit just before substituting 0 for h.

RC2.4 determine, through investigation using technology, the graph of the derivative f'(x)or dy of a given sinusoidal function [i.e., dx f(x) = sin x, f(x) = cos x] (e.g., by generating a table of values showing the instantaneous rate of change of the function for various values of x and graphing the ordered pairs; by using dynamic geometry software to verify graphically that when f(x) = sin x, f'(x) = cos x, and when f(x) = cos x, f'(x) = –sin x; by using a motion sensor to compare the displacement and velocity of a pendulum)

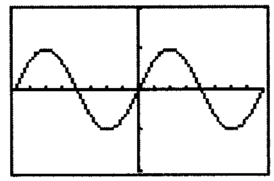
The Derivatives of Sinusoidal Functions

The functions $f(x) = \sin x$ and $f(x) = \cos x$ are both sinusoidal functions, each with a period of 2π. It follows that the derivatives of these functions will also be sinusoidal, each with a period of 2π. As with other functions, the graphs of sinusoidal functions and their derivatives can be investigated by using a table of values or graphing technology in which the following can be determined.

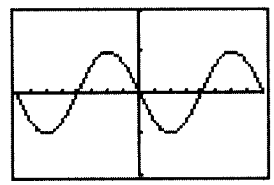
- If $f(x) = \sin x$, then $f'(x) = \cos x$.
- If $f(x) = \cos x$, then $f'(x) = -\sin x$.

18. Which of the following calculator screen shots (each using the window $x:\left[-2\pi, 2\pi, \frac{\pi}{4}\right] y:[-2, 2, 1]$) corresponds to the graph of the derivative of the sinusoidal function $f(x) = -\sin x$?

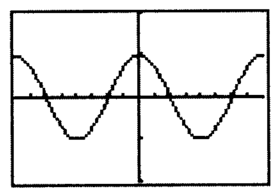
A.



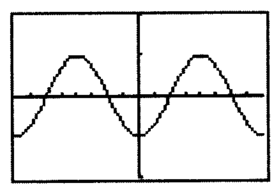
B.



C.



D.

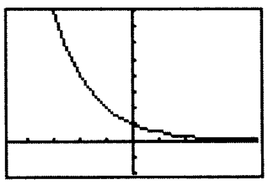


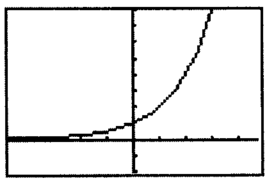
RC2.5 determine, through investigation using technology, the graph of the derivative f'(x) or dy/dx of a given exponential function [i.e., f(x) = a(x) (a > 0, a ≠ 1)], and make connections between the graphs of f(x) and f'(x) or y and dy/dx

Analyzing the Derivatives of Exponential Functions Graphically

An exponential function is a function of the form $f(x) = a^x$, where $a > 0$ and $a \neq 1$.

The graphs for $f(x) = \left(\frac{1}{2}\right)^x$ and $f(x) = 2^x$, from a TI-83 graphing calculator, are shown.





When $0 < a < 1$, from left to right the graph falls toward the x-axis, and the derivative of the function will always be negative since it is always decreasing. When $a > 1$, from left to right the graph rises from near the x-axis, and the derivative of the function will always be positive since it is always increasing.

The derivative of an exponential function is related to the function itself by $f'(x) = kf(x)$, where k is a constant, and the value of k varies with the value of a.

19. Given the function $f(x) = 5^{3x}$, the ratio $\frac{f'(x)}{f(x)}$, to the nearest thousandth, is

A. 0.207 **B.** 0.536

C. 1.864 **D.** 4.828

Written Response

20. Using technology, generate a table of values that shows the instantaneous rate of change of the function $f(x) = \left(\frac{1}{4}\right)^x$ on the interval $-4 \leq x \leq 4, x \in I$.

RC2.6 determine, through investigation using technology, the exponential function f(x) = a(x) (a > 0, a ≠ 1) for which f'(x) = f(x) (e.g., by using graphing technology to create a slider that varies the value of a in order to determine the exponential function whose graph is the same as the graph of its derivative), identify the number e to be the value of a for which f'(x) = f(x) [i.e., given f(x) = e(x), f(x) = e(x)], and recognize that for the exponential function f(x) = e(x) the slope of the tangent at any point on the function is equal to the value of the function at that point

Analyzing the Derivatives of Exponential Functions When $f'(x) = f(x)$

For functions of the form $f(x) = a^x$, the derivative function is of the form $f'(x) = kf(x)$, where k is a constant. Also, the value of k varies with the value of a.

One approach to finding a value of a for the function $f(x) = a^x (a > 0, a \neq 1)$ so that $f'(x) = f(x)$ (the value of k is 1) is to plot the graphs of $y = f(x) = a^x$ and its derivative $y = f'(x)$ for various values of a until the graphs appear to overlap.

After enough trials, the graphs will overlap at a number known as Euler's number, *e*.
The number *e* is an irrational number. When written as a decimal, it neither terminates nor repeats.
The first 13 decimal places are 2.718 281 8284 (59).
The derivative of $f(x) = e^x$ is $f'(x) = e^x$. It is the only function whose derivative is the function itself.
This means that the instantaneous rate of change of $f(x) = e^x$ at any value of x is e^x, which is also the slope of the tangent line of the graph at that particular x-value.

Use the following information to answer the next question.

The slope of the tangent for the function $f(x) = a^x$ is 1 at the point (0, 1).

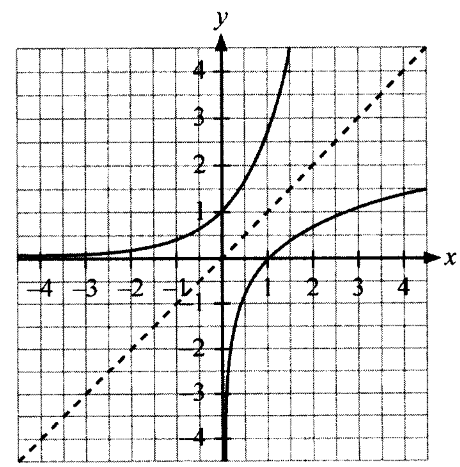
Numerical Response

21. Calculated to the nearest hundredth, the value of a is __________.

RC2.7 recognize that the natural logarithmic function f(x) = log(e)x, also written as f(x) = ln x, is the inverse of the exponential function f(x) = e(x), and make connections between f(x) = ln x and f(x) = e(x)

Investigating the Natural Logarithmic Function $f(x) = \ln x$

A **natural logarithm** is any logarithm with a base of the unique numeric value *e*. The value of *e* is unique because the function of $f(x) = e^x$ has the derivative of $f'(x) = e^x$. The inverse of $f(x) = e^x$ is the natural logarithmic function, $f(x) = \log_e x$. The notation ln, pronounced "lawn," is used to represent the natural logarithm $\log_e$. Therefore, the function $f(x) = \log_e x$ is written as $f(x) = \ln x$. The graphs of $y = e^x$ and $y = \ln x$ will appear as reflections across the line $y = x$ since two inverse relations are graphed as reflections in the line $y = x$.



The functions of $y = e^x$ and $y = \ln x$

Since $f(x) = e^x$ and $f(x) = \ln x$ are inverse functions, the operation of one undoes the operation of the other. Consider the composite function $g(x) = e^{\ln x}$. Let $x = 20$ therefore $g(20) = e^{\ln 20}$. The value of ln 20 is approximately 3.0, and the value of $e^{\approx 3.0}$ is 20. Therefore, $g(20) = e^{\ln 20} = 20$.

Generally, $e^{\ln x} = x$ and $\ln e^x = x$.

22. What is the value of x in the equation $e^{4-x} = 12$?

A. $x = e^{-8}$

B. $x = -8$

C. $x = \ln(12) - 4$

D. $x = 4 - \ln(12)$

RC2.8 verify, using technology (e.g., calculator, graphing technology), that the derivative of the exponential function f(x) = a(x) is f'(x) = ax ln a for various values of a

Investigating the Derivative of Exponential Functions Algebraically

Given an exponential function of the form $f(x) = a^x$, the derivative is related to the function itself by $f'(x) = kf(x)$, where k is a constant, and the value of k varies with the value of a. By building a table of values and comparing the value of $a^x \ln(a)$ to the expression, $\frac{f(x+h) - f(x)}{h}$, where x is any value and h is a value that is approaching 0, it can be verified that the value of k is equal to $\ln a$. Therefore, if $f(x) = a^x$, then the derivative of the function is $f'(x) = a^x \ln(a)$.

23. The value of $f'(1.5)$ for the function $f(x) = 4^x$, to the nearest hundredth, is

A. 0.41 **B.** 1.39

C. 8.50 **D.** 11.09

RC3.1 verify the power rule for functions of the form f(x) = x(n), where n is a natural number

The Power Rule on Functions of the Form $f(x) = x^n$, Where n is a Natural Number

The derivative of a polynomial function of degree n is of degree $n - 1$. The derivatives of polynomial functions of the form $f(x) = x^n$, where n is a natural number, have a specific form that is described by the power rule.

The power rule states that the derivative of $f(x) = x^n$, where $n \in N$, is $f'(x) = nx^{n-1}$.

24. The derivative of $y = t^{k+1}$, $k \in \mathbb{N}$, gives a result of

A. $\frac{dy}{dt} = (k+1)t^k$

B. $\frac{dy}{dt} = (k-1)t^k$

C. $\frac{dy}{dt} = kt^{k-1}$

D. $\frac{dy}{dt} = kt^k$

RC3.2 verify the constant, constant multiple, sum, and difference rules graphically and numerically, and read and interpret proofs involving lim[h arrow 0] f(x + h) – f(x)/h of the constant, constant multiple, sum, and difference rules (student reproduction of the development of the general case is not required)

The Constant, Constant Multiple, Sum, and Difference Rules

When taking the derivative of a function, the following rules should be considered:

- The constant rule—the derivative of $f(x) = k$, where $k \in \mathbb{R}$, is $f'(x) = 0$.
- The constant multiple rule—the derivative of $g(x) = kf(x)$, where $k \in \mathbb{R}$, is $g'(x) = kf'(x)$.
- The sum rule—$f'(x) + g'(x) = (f+g)'(x)$.
- The difference rule—$f'(x) - g'(x) = (f-g)'(x)$.

Recognizing and interpreting proofs of the derivative rules can be done using the definition of the derivative, $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

25. Which rule is illustrated by the statement that the derivative of $f(x) = 7$ is $f'(x) = 0$?

A. Sum rule

B. Constant rule

C. Difference rule

D. Constant multiple rule

Written Response

26. Explain and illustrate how the difference rule for derivatives could be verified graphically and numerically for the functions $f(x) = 3x^2$ and $g(x) = x^3$.

RC3.3 determine algebraically the derivatives of polynomial functions, and use these derivatives to determine the instantaneous rate of change at a point and to determine point(s) at which a given rate of change occurs

Using Derivatives of Polynomials to Determine an Instantaneous Rate of Change Algebraically

The power, constant, constant multiple, sum, and difference rules can be applied to finding the derivatives of many polynomial functions. Once the derivative of a polynomial function, $f(x)$, is determined, it can be used to do the following:

- The instantaneous rate of change of a function, $f(x)$, at a point can be determined by substituting the value of the x-coordinate into the derivative and solving for $f'(x)$.
- The point or points where a function, $f(x)$, changes at a particular rate can be determined by substituting the rate of change for $f'(x)$ in the derivative and solving for x.

27. The instantaneous rate of change of the function $f(x) = x^4 - 11x^2 + 7x - 1$ at $x = 0$ is

A. $f'(0) = -1$
B. $f'(0) = 0$
C. $f'(0) = 2$
D. $f'(0) = 7$

28. For the function $f(x) = -3x^5 + 12x^2 - 15$, $f'(2)$ is equal to

A. -63 **B.** -129
C. -192 **D.** -288

Numerical Response

29. Given the function $f(x) = x^2 - 3$, the y-coordinate of the highest point on the graphs of $f(x)$ and $f'(x)$ that satisfies the equation $f(x) = f'(x)$ is _________.

Written Response

30. Find the x-coordinates of the points on the curve of $f(x) = x^3 - x^2 - x + 1$ where the tangent line to $f(x)$ is horizontal.

RC3.4 verify that the power rule applies to functions of the form f(x) = x(n), where n is a rational number, and verify algebraically the chain rule using monomial functions and the product rule using polynomial functions

The Power, Chain, and Product Rules

The power, chain, and product rules can be applied to determine the derivatives of various functions. Each rule can be verified by comparing the derivatives of functions that were determined by using different methods.

The Power Rule

The power rule states that the derivative of $f(x) = x^n$ is $f'(x) = nx^{n-1}$. It applies when n is a rational number (a number of the form $\frac{A}{B}$, where A and B are integers and $B \neq 0$). To verify the power rule, the values of the slopes of tangents to a function $f(x)$ can be compared to the values of its derivative function $f'(x)$ found using the power rule.

The Chain Rule

For functions that are of the form $f(x) = g(h(x))$, the derivative is often found by using the chain rule. The derivative of $f(x) = g(h(x))$ is $f'(x) = g'(h(x))h'(x)$.

Applying the chain rule to a function of the form $f(x) = (h(x))^n$ gives a derivative of $f'(x) = n(h(x))^{n-1}h'(x)$. If the function $h(x)$ is a monomial, the chain rule can be verified by finding the same derivative from a simplified form of $(h(x))^n$.

The Product Rule

Derivatives of functions of the form $f(x) = g(x)h(x)$ are often found by using the product rule. The derivative of $f(x) = g(x)h(x)$ is $f'(x) = g(x)h'(x) + g'(x)h(x)$.

The product rule can be abbreviated as $(gh)' = gh' + g'h$.

If a function is a product of other functions, the product rule can be verified by finding the derivative of a simplified form of the product.

31. Which of the following functions could be used to verify the product rule for finding the derivative of $f(x) = (3x^2 - 3)(5x^3 + 2x)$?

A. $f(x) = 75x^4 - 6$

B. $f(x) = 15x^5 - 6x$

C. $f(x) = 15x^5 - 9x^3 - 6x$

D. $f(x) = 75x^4 - 27x^2 - 6$

*RC3.5 solve problems, using the product and chain rules, involving the derivatives of polynomial functions, sinusoidal functions, exponential functions, rational functions, radical functions, and other simple combinations of functions**

Applying the Product and Chain Rules

The product and chain rules can be used to determine the derivatives of a variety of polynomial and non-polynomial functions. To apply the chain rule, rewrite a function as a power function.
To apply the product rule, change the form of a function from a quotient to a product. The latter type involves the application of both rules.

Example

Determine $\frac{dy}{dx}$ if $y = \sqrt{x^2 - 4x}$.

Solution

Step 1
Change the radical to a power.

$$\begin{aligned} y &= \sqrt{x^2 - 4x} \\ &= (x^2 - 4x)^{\frac{1}{2}} \end{aligned}$$

Step 2
Apply the chain rule.

$$\begin{aligned} y &= (x^2 - 4x)^{\frac{1}{2}} \\ \frac{dy}{dx} &= \frac{1}{2}(x^2 - 4x)^{-\frac{1}{2}}\left(\frac{d}{dx}(x^2 - 4x)\right) \\ &= \frac{1}{2}(x^2 - 4x)^{-\frac{1}{2}}(2x - 4) \\ &= \frac{2x - 4}{2(x^2 - 4x)^{\frac{1}{2}}} \\ &= \frac{x - 2}{(x^2 - 4x)^{\frac{1}{2}}} \\ &= \frac{x - 2}{\sqrt{x^2 - 4x}} \end{aligned}$$

Example

Determine the derivative of $f(x) = \frac{x-2}{x^2+2}$.

Solution

Step 1

Change the function from quotient form to product form.

$$f(x) = \frac{x-2}{x^2+2}$$
$$= (x-2)(x^2+2)^{-1}$$

Step 2

Let $g(x) = x - 2$ and $h(x) = (x^2+2)^{-1}$, and determine $g'(x)$ and $h'(x)$.

$$g(x) = x - 2$$
$$g'(x) = 1$$
$$h(x) = (x^2+2)^{-1}$$
$$h'(x) = -2x(x^2+2)^{-2}$$

Step 3

Apply the product rule to determine $f'(x)$.

$$f(x) = g(x)h(x)$$
$$f'(x) = g'(x)h(x) + g(x)h'(x)$$
$$= \left((x^2+2)^{-1} + (x-2)\left(-2x(x^2+2)^{-2}\right) \right)$$
$$= \frac{1}{x^2+2} - \frac{2x(x-2)}{(x^2+2)^2}$$

---

32. The derivative $\frac{dy}{dx}$ of $y = -3x^3e^{2x}$ is

A. $-18x^2e^{2x}$

B. $-3x^3e^{2x} - 9x^2e^{2x}$

C. $-6x^3e^{2x} - 9x^2e^{2x}$

D. $-3x^3e^2 - 9x^2e^{2x}$

Numerical Response

33. To the nearest tenth, what is the value of $f'(1)$ given $f(x) = \frac{(7x-3)^2}{3x^3}$? ________

Written Response

34. Determine the equation of the tangent line to the curve defined by $y = (1+x^3)^{\frac{3}{2}}$ at the point (2, 27).

Written Response

35. Determine the derivative of $y = \frac{e^{3x}}{x^3}$.

Use the following information to answer the next question.

The functions t and u are differentiable at $x = 3$.

Written Response

36. If $t(3) = 4$, $t'(3) = 5$, $u(3) = -2$, and $u'(3) = -1$, determine the value of the derivative of tu when $x = 3$.

Answers and Solutions

Rate of Change

1. B	9. WR	17. WR	25. B	33. 2.7
2. B	10. A	18. D	26. WR	34. WR
3. C	11. WR	19. D	27. D	35. WR
4. WR	12. B	20. WR	28. C	36. WR
5. D	13. WR	21. 2.72	29. 6	
6. 24	14. WR	22. D	30. WR	
7. D	15. A	23. D	31. C	
8. WR	16. B	24. A	32. C	

1. B

A positive rate of change means that as an independent variable increases, the dependent variable increases. Consider each of the given variables individually.

The temperature of water being heated over time increases as the time of heating increases.

The tension of a spring increases as the amount of compression increases.

The average monthly temperature in Canada increases from January to May (as the months of the year increase).

The fuel economy of a vehicle decreases as a vehicle increases its speed. This does not represent a positive rate of change because as the independent variable (speed) increases, the dependent variable (fuel economy) decreases.

2. B

Step 1
Determine the position, s, at $t = 0$.

$$\begin{aligned} s &= 2t - 5 \\ &= 2(0) - 5 \\ &= -5 \text{ m} \end{aligned}$$

Step 2
Determine the position, s, at $t = 5$.

$$\begin{aligned} s &= 2t - 5 \\ &= 2(5) - 5 \\ &= 5 \text{ m} \end{aligned}$$

Step 3
Determine the rate of change of the object.
Since the object moves from −5 m to 5 m (a total of 10 m) in 5 s, the rate of change is $\frac{10 \text{ m}}{5 \text{ s}} = 2.0$ m/s.

3. C

The average rate of change is the slope, m, between the points $(-1, 6)$ and $(-5, 4)$.

$$\begin{aligned} m &= \frac{4-6}{-5-(-1)} \\ &= \frac{-2}{-4} \\ &= \frac{1}{2} \end{aligned}$$

Therefore, the average rate of change is $\frac{1}{2}$ units of y for every unit of x.

4. WR

The quotient $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ represents the slope of the secant line through the points $(x_2, f(x_2))$ and $(x_1, f(x_1))$. It also represents the average rate of change of the function $y = f(x)$ over the interval from x_1 to x_2.

5. D

To best approximate the instantaneous rate of change of the function at $x = -3$, choose points close to $x = -3$ on both sides of the point.

The closest pair of points (excluding the point at $x = -3$) among the given pairs of points is $(-3.5, -11.31)$ and $(-2.5, -5.657)$.

6. 24

Use technology or a table of values to determine the instantaneous rate of change at $x = 2$ by finding the slopes of secant lines passing through the point $x = 2$ and points with x-values close by, such as $x = 1.99$, $x = 2.01$, $x = 1.999$, and $x = 2.001$.

Note that the point at $x = 2$ is $\left(2, 2(2)^3 - 5\right) = (2, 11)$.

Value of x	Point	Slope of Secant
1.99	(1.99, 10.7612)	$\frac{10.7612 - 11}{1.99 - 2}$ $= 23.88$
2.01	(2.01, 11.2412)	$\frac{11.2412 - 11}{2.01 - 2}$ $= 24.12$
1.999	(1.999, 10.9760)	$\frac{10.9760 - 11}{1.999 - 2}$ $= 24$
2.001	(2.001, 11.024)	$\frac{11.024 - 11}{2.001 - 2}$ $= 24$

According to the table of values, the slope of the secant line approaches a value of 24. Therefore, the instantaneous rate of change at $x = 2$ is 24.

7. D

The table of values shows that there is no vertical asymptote or hole at $x = 0$, since the point $(0, -1)$ exists. As x approaches larger positive values, $f(x)$ approaches 0, which implies a horizontal asymptote at $y = 0$ and no x-intercept.

8. WR

The expression $\frac{f(a+h) - f(a)}{h}$ can be used for calculating average rates of change; for example, the average speed of a vehicle between $t = a$ and $t = a + h$. The expression $\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ can be used for calculating instantaneous rates of change; for example, the instantaneous speed of a vehicle at $t = a$.

9. WR

The instantaneous rate of change at $x = a$ for a function, f, can be determined using the expression $\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

In the expression $\lim\limits_{h \to 0} \frac{\frac{2}{(-4+h)^2} - \frac{2}{(-4)^2}}{h}$, $f(a) = \frac{2}{(-4)^2}$, when $a = -4$. Therefore, when $x = a$, the function is $f(x) = \frac{2}{x^2}$.

10. A

For each function, calculate the expression $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ for $x = -7$.

Step 1

$f(x) = x^3$

$$\lim_{h \to 0} \frac{(-7+h)^3 - (-7)^3}{h}$$
$$= \lim_{h \to 0} \frac{(49 - 14h + h^2)(-7+h) - (-343)}{h}$$
$$= \lim_{h \to 0} \frac{(-343 + 147h - 21h^2 + h^3) + 343}{h}$$
$$= \lim_{h \to 0} \frac{147h - 21h^2 + h^3}{h}$$
$$= \lim_{h \to 0} \frac{h(147 - 21h + h^2)}{h}$$
$$= 147$$

Step 2

$f(x) = x^3 - x$

$$\lim_{h \to 0} \frac{((-7+h)^3 - (-7+h)) - ((-7)^3 - (-7))}{h}$$
$$= \lim_{h \to 0} \frac{(-343 + 147h - 21h^2 + h^3 + 7 - h) - (-343 + 7)}{h}$$
$$= \lim_{h \to 0} \frac{146h - 21h^2 + h^3}{h}$$
$$= \lim_{h \to 0} \frac{h(146 - 21h + h^2)}{h}$$
$$= 146$$

Step 3

$f(x) = x^2 + 9$

$$\lim_{h \to 0} \frac{((-7+h)^2 + 9) - ((-7)^2 + 9)}{h}$$
$$= \lim_{h \to 0} \frac{(49 - 14h + h^2 + 9) - (49 + 9)}{h}$$
$$= \lim_{h \to 0} \frac{-14h + h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(-14 + h)}{h}$$
$$= -14$$

Step 4

$f(x) = 2x^2 + 3x$

$$\lim_{h \to 0} \frac{\left(2(-7+h)^2 + 3(-7+h)\right) - \left(2(-7)^2 + 3(-7)\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(2\left(49 - 14h + h^2\right) - 21 + 3h\right) - (98 - 21)}{h}$$

$$= \lim_{h \to 0} \frac{\left(98 - 28h + 2h^2 - 21 + 3h\right) - (77)}{h}$$

$$= \lim_{h \to 0} \frac{-25h + 2h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(-25 + 2h)}{h}$$

$$= -25$$

11. WR

Simplify $\frac{f(a+h) - f(a)}{h}$ for $f(x) = x^3 - 1$, where $a = -5$.

$$\frac{f(a+h) - f(a)}{h}$$

$$= \frac{f(-5+h) - f(-5)}{h}$$

$$= \frac{\left((-5+h)^3 - 1\right) - \left((-5)^3 - 1\right)}{h}$$

$$= \frac{\left((-5+h)\left(25 - 10 + h^2\right) - 1\right) - (-125 - 1)}{h}$$

$$= \frac{\left(-125 + 75h - 15h^2 + h^3 - 1\right) - (-126)}{h}$$

$$= \frac{75h - 15h^2 + h^3}{h}$$

$$= \frac{h\left(75 - 15h + h^2\right)}{h}$$

$$= 75 - 15h + h^2$$

$$= h^2 - 15h + 75$$

12. B

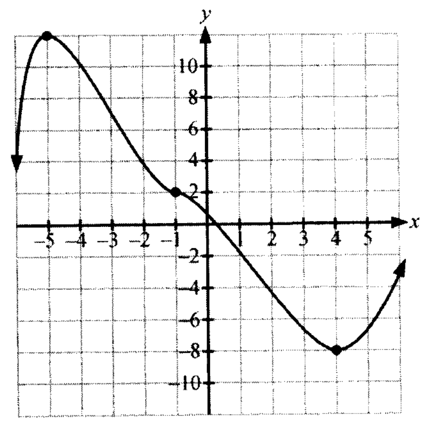
For the function to have slopes of tangents that are positive for $x < -3$ and negative for $-3 < x < 1$, there must be a local maximum at $x = -3$.

For the slopes of tangents to go from negative for $-3 < x < 1$ to positive for $x > 1$, there must be a local minimum at $x = 1$. Therefore, the maximum number of minima is 1.

13. WR

There is a local maximum at $x = -5$ because the instantaneous rate of change is equal to 0 at $x = -5$ while the rate of change is positive on the interval $x < -5$ and negative on the interval $-5 < x < -1$. Similarly, there is a local minimum at $x = 4$ because the instantaneous rate of change is equal to 0 at $x = 4$ while the rate of change is negative on $-1 < x < 4$ and positive on $x > 4$. There is also a turning point at $x = -1$ since the slope of the graph is 0 at $x = -1$ while being negative on $-5 < x < -1$ and $-1 < x < 4$.

The graph drawn should resemble the one shown here.



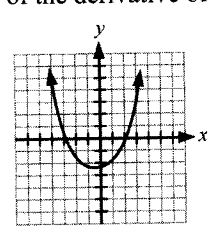
14. WR

The rate of change is greatest for $x > 5$ because the slopes of the tangents on this interval are the greatest. The slopes are positive and very large because the graph is steepest on this interval.

15. A

At approximately $x = -3$ and $x = 2$, the slopes of the tangents of $y = f(x)$ are 0. Therefore, the graph of the derivative is equal to zero at $x = -3$ and $x = 2$. The slopes of the tangents of $y = f(x)$ in the intervals $x < -3$ and $x > 2$ are positive, and the slopes of the tangents in the interval $-3 < x < 2$ are negative. Therefore, the graph of the derivative is positive in the intervals $x < -3$ and $x > 2$ and negative in the interval $-3 < x < 2$.

This graph fits these descriptions, so it is the graph of the derivative of the function $y = f(x)$.



Also, since the original polynomial function looks cubic, the derivative function should look quadratic.

16. B

The derivative of a function, $f(x)$, is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

To determine $f'(0)$, substitute $x = 0$ and simplify the limit for $f(x) = -12x + 9$.

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \to 0} \frac{(-12(0+h)+9) - (-12(0)+9)}{h} \\ &= \lim_{h \to 0} \frac{(-12h+9) - (9)}{h} \end{aligned}$$

The simplified expression for $f'(0)$ is $\lim_{h \to 0} \frac{(-12h+9) - 9}{h}$.

17. WR

Simplify the definition of the derivative for $f(x) = x^2 + 5x - 11$.

$$\begin{aligned} &f'(x) \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{((x+h)^2 + 5(x+h) - 11) - (x^2 + 5x - 11)}{h} \\ &= \lim_{h \to 0} \frac{x^2 + 2xh + h^2 + 5x + 5h - 11 - x^2 - 5x + 11}{h} \\ &= \lim_{h \to 0} \frac{2xh + h^2 + 5h}{h} \\ &= \lim_{h \to 0} \frac{h(2x + h + 5)}{h} \\ &= \lim_{h \to 0} 2x + h + h \end{aligned}$$

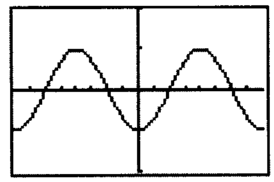
The simplified expression in the numerator of the limit is $2x + h + 5$.

18. D

The derivative of $f(x) = \sin(x)$ is $f'(x) = \cos(x)$. Therefore, the derivative of $f(x) = -\sin(x)$ is $f'(x) = -\cos(x)$.

Verify using a graphing calculator by entering in the function and its derivative, $Y_1 = -\sin(X)$ and $Y_2 = \text{nDeriv}(Y_1, X, X)$, as y-values and pressing [GRAPH].

The following screen shot corresponds to the graph of the derivative of the sinusoidal function $f(x) = -\sin x$:



19. D

On a graphing calculator, enter the function, its derivative, and the quotient $\frac{f'(x)}{f(x)}$ as y-values.

$Y_1 = 5$^$(3X)$
$Y_2 = \text{nDeriv}(Y_1, X, X)$
$Y_3 = Y_2/Y_1$

Adjust the window settings to x:[−2, 2, 1] and y:[−1, 8, 1]. Press [GRAPH], and use the trace function to determine the ratio $\frac{f'(x)}{f(x)}$.

$$\begin{aligned} Y_3 &= \frac{Y_2}{Y_1} \\ &\approx 4.828 \end{aligned}$$

20. WR

Step 1
Clear all entries in the table of a TI-83 or similar calculator.

Press [2nd] [0], scroll down to ClrTable, and press [ENTER] twice.

Step 2
Enter the given function into the [Y =] editor of the calculator.

Press [Y =], enter $\left(\frac{1}{4}\right)^x$ in Y_1, and press [ENTER].

Step 3
Enter the derivative of the given function in Y_2.

Press [MATH], and select 8:nDeriv(. Press [VARS], and scroll right to select Y-VARS. Select 1:Function... and then 1:Y_1. Press [,] to insert a comma, and then enter X, X).

Step 4
Create a table that shows x, $f(x)$, and the instantaneous rate of change of the function on the interval $-4 \le x \le 4$, $x \in I$.

Press 2nd WINDOW to bring up the TBLSET menu. Enter −4 in TblStart= and 1 in ΔTbl=. Then, press 2nd GRAPH. Scroll down to see all the values in the given table.

x	$f(x)$	Rate of Change
−4	256	− 354.9
−3	64	−88.72
−2	16	−22.18
−1	4	−5.545
0	1	−1.386
1	0.25	−0.3466
2	0.0625	−0.0866
3	0.015 63	−0.0217
4	0.003 91	−0.0054

21. 2.72

The slope of the tangent at any point for $f(x) = e^x$ is equal to the value of the function at that point, since the derivative of $f(x) = e^x$ is $f'(x) = e^x$. It is the only function whose derivative is the function itself. Therefore, if the slope at (0, 1) is 1, the function must be $f(x) = e^x$, where $e \approx 2.72$ to the nearest hundredth.

22. D

Step 1
Take the natural logarithm of both sides of the equation.

$$e^{4-x} = 12$$
$$\ln\left(e^{4-x}\right) = \ln(12)$$

Step 2
Simplify the equation.

Since $\ln\left(e^x\right) = x$, then $\ln\left(e^{4-x}\right) = 4 - x$. Substitute $4 - x$ for $\ln\left(e^{4-x}\right)$ in the equation and isolate x.

$$\ln\left(e^{4-x}\right) = \ln(12)$$
$$4 - x = \ln(12)$$
$$x = 4 - \ln(12)$$

23. D

The derivative of $f(x) = a^x$ is $f'(x) = a^x \ln(a)$. Therefore, $f'(1.5)$ for the function $f(x) = 4^x$ can be calculated as follows:

$$f(x) = 4^x$$
$$f'(x) = 4^x \ln(4)$$
$$f'(1.5) = 4^{1.5} \ln(4)$$
$$f'(1.5) \approx 8(1.386\,294\,361)$$
$$f'(1.5) \approx 11.09$$

24. A

The power rule states that the derivative of $f(x) = x^n$, $n \in \mathbb{N}$, is $f'(x) = nx^{n-1}$. Therefore, applying the power rule will give the following result:

$$y = t^{k+1}$$
$$\frac{dy}{dx} = (k+1)t^{(k+1)-1}$$
$$\frac{dy}{dx} = (k+1)t^k$$

25. B

The constant rule states that the derivative of $f(x) = k$, where $k \in \mathbb{R}$, is $f'(x) = 0$.

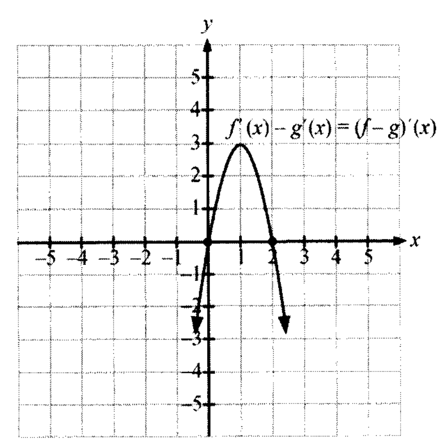
The function $f(x) = 7$ has a k-value of 7, which is a value in the real number domain, and has a derivative of $f'(x) = 0$.

Therefore, the constant rule is being illustrated by the statement that the derivative of $f(x) = 7$ is $f'(x) = 0$.

26. WR

Graphically, verification would show that the graphs of $f'(x) - g'(x)$ and $(f - g)'(x)$ are identical. Numerically, verification could come in the form of a table of values in which $f'(x) - g'(x)$ is shown to be equal to $(f - g)'(x)$ for the chosen values of x.

Verified graphically:



Verified numerically:
Some possible values for verification are shown in the given table.

x	$f'(x) = 6x$	$g'(x) = 3x^2$	$f'(x) - g'(x) = 6x - 3x^2$	$(f - g)'(x) = 6x - 3x^2$
–3	–18	27	–45	–45
–2	–12	12	–24	–24
–1	–6	3	–9	–9
0	0	0	0	0
1	6	3	3	3
2	12	12	0	0
3	18	27	–9	–9

27. D

Step 1
Apply the power rule to determine the derivative of the function $f(x) = x^4 - 11x^2 + 7x - 1$.

$$f(x) = x^4 - 11x^2 + 7x - 1$$
$$f'(x) = 4x^3 - 22x + 7$$

Step 2
Substitute 0 for x in the equation $f'(x) = 4x^3 - 22x + 7$, and solve for $f'(0)$ to determine the instantaneous rate of change at $x = 0$.

$$f'(x) = 4x^3 - 22x + 7$$
$$f'(0) = 4(0)^3 - 22(0) + 7$$
$$= 7$$

28. C

Step 1
Apply the power rule to determine the derivative of $f(x) = -3x^5 + 12x^2 - 15$.

$$f(x) = -3x^5 + 12x^2 - 15$$
$$f'(x) = -15x^4 + 24x$$

Step 2
Substitute 2 for x in the equation $f'(x) = -15x^4 + 24x$, and solve for $f'(2)$.

$$f'(x) = -15x^4 + 24x$$
$$f'(2) = -15(2)^4 + 24(2)$$
$$= -240 + 48$$
$$= -192$$

29. 6

Step 1
Apply the power rule to determine the derivative of the function $f(x) = x^2 - 3$.

$$f(x) = x^2 - 3$$
$$f'(x) = 2x$$

Step 2
Equate $f(x)$ with $f'(x)$, and solve for x.

$$f(x) = f'(x)$$
$$x^2 - 3 = 2x$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$

Therefore, $x = 3$ and $x = -1$.

Step 3
Substitute $x = 3$ into the equation $f(x) = x^2 - 3$, and solve for $f(3)$.

$$f(x) = x^2 - 3$$
$$f(3) = (3)^2 - 3$$
$$= 9 - 3$$
$$= 6$$

Step 4
Substitute $x = -1$ into the equation $f(x) = x^2 - 3$, and solve for $f(-1)$.

$$f(x) = x^2 - 3$$
$$f(-1) = (-1)^2 - 3$$
$$= 1 - 3$$
$$= -2$$

The points (3, 6) and (–1, –2) both satisfy the equation $f(x) = f'(x)$. The y-coordinate of the highest point is $y = 6$.

30. WR

When the tangent line of $f(x)$ is horizontal, $f'(x) = 0$.

Step 1
Apply the power law to determine the derivative of the function $f(x) = x^3 - x^2 - x + 1$.

$$f(x) = x^3 - x^2 - x + 1$$
$$f'(x) = 3x^2 - 2x - 1$$

Step 2
Substitute 0 for $f'(x)$ into the equation $f'(x) = 3x^2 - 2x - 1$, and solve for x.

$$f'(x) = 3x^2 - 2x - 1$$
$$0 = 3x^2 - 2x - 1$$
$$0 = 3x^2 - 3x + x - 1$$
$$0 = 3x(x - 1) + (x - 1)$$
$$0 = (3x + 1)(x - 1)$$

Therefore, $x = -\frac{1}{3}$ and $x = 1$.

31. C

Simplify the function $f(x) = (3x^2 - 3)(5x^3 + 2x)$.

$$f(x) = (3x^2 - 3)(5x^3 + 2x)$$
$$= 15x^5 + 6x^3 - 15x^3 - 6x$$
$$= 15x^5 - 9x^3 - 6x$$

Therefore, the function $f(x) = 15x^5 - 9x^3 - 6x$ could be used to verify the product rule.

32. C

Let $g(x) = -3x^3$ and $h(x) = e^{2x}$.

Step 1
Determine $g'(x)$ and $h'(x)$.

$$g(x) = -3x^3$$
$$g'(x) = -9x^2$$
$$h(x) = e^{2x}$$
$$h'(x) = e^{2x}\left(\frac{d}{dx}2x\right)$$
$$= 2e^{2x}$$

Step 2
Determine the derivative of the equation $y = -3x^3e^{2x}$ by applying the product rule.

$$y = g(x)h(x)$$
$$\frac{dy}{dx} = g'(x)h(x) + g(x)h'(x)$$
$$= (-9x^2)(e^{2x}) + (-3x^3)(2e^{2x})$$
$$= -9x^2e^{2x} - 6x^3e^{2x}$$
$$= -6x^3e^{2x} - 9x^2e^{2x}$$

33. 2.7

Step 1
Rewrite the function in product form.

$$f(x) = \frac{(7x-3)^2}{3x^3}$$
$$= (7x-3)^2(3x^3)^{-1}$$
$$= (7x-3)^2\left(\frac{1}{3}x^{-3}\right)$$

Let $g(x) = (7x-3)^2$and $h(x) = \frac{1}{3}x^{-3}$.

Step 2
Determine $g'(x)$ and $h'(x)$.

$$g(x) = (7x-3)^2$$
$$g'(x) = 2(7x-3)\left(\frac{d}{dx}(7x-3)\right)$$
$$= 2(7x-3)(7)$$
$$= 98x - 42$$
$$h(x) = \frac{1}{3}x^{-3}$$
$$h'(x) = -x^{-4}$$

Step 3
Apply the product rule to find $f'(x)$.

$$f(x) = g(x)h(x)$$
$$f'(x) = g'(x)h(x) + g(x)h'(x)$$
$$= \begin{pmatrix} (98x-42)\left(\frac{1}{3}x^{-3}\right) \\ + (7x-3)^2(-x^{-4}) \end{pmatrix}$$
$$= \begin{pmatrix} 32.\overline{66}x^{-2} - 14x^{-3} \\ - 49x^{-2} + 42x^{-3} - 9x^{-4} \end{pmatrix}$$
$$= -16.\overline{33}x^{-2} + 28x^{-3} - 9x^{-4}$$

Step 4
Substitute 1 for x into the equation $f'(x) = -16.\overline{33}x^{-2} + 28x^{-3} - 9x^{-4}$, and solve for $f'(1)$.

$$f'(x) = -16.\overline{33}x^{-2} + 28x^{-3} - 9x^{-4}$$
$$f'(1) = -16.\overline{33}(1)^{-2} + 28(1)^{-3} - 9(1)^{-4}$$
$$= -16.\overline{33} + 28 - 9$$
$$= 2.\overline{66}$$
$$\approx 2.7$$

34. WR

Step 1
Apply the chain rule to determine the derivative of the equation $y = (1+x^3)^{\frac{3}{2}}$.

$$y = (1+x^3)^{\frac{3}{2}}$$
$$\frac{dy}{dx} = \frac{3}{2}(1+x^3)^{\frac{1}{2}}\left(\frac{d}{dx}(1+x^3)\right)$$
$$\frac{dy}{dx} = \frac{3}{2}(1+x^3)^{\frac{1}{2}}(3x^2)$$
$$\frac{dy}{dx} = \frac{9}{2}x^2\sqrt{1+x^3}$$

Step 2
Find the slope of the tangent line at (2, 27).

Solve for $\frac{dy}{dx}$ when $x = 2$.

$$\frac{dy}{dx} = \frac{9}{2}x^2\sqrt{1+x^3}$$
$$= \frac{9}{2}(2)^2\sqrt{1+(2)^3}$$
$$= \frac{9}{2}(4)\sqrt{1+8}$$
$$= 18\sqrt{9}$$
$$= 18(3)$$
$$= 54$$

Step 3
Substitute 2 for x, 27 for y, and 54 for m into the equation of a line formula $y = mx + b$, and solve for b.

$$\begin{aligned} y &= mx + b \\ 27 &= 54(2) + b \\ 27 &= 108 + b \\ -81 &= b \end{aligned}$$

The equation of the tangent line that goes through the point (2, 27) is $y = 54x - 81$.

35. WR

Step 1

Convert the equation $y = \dfrac{e^{3x}}{x^3}$ into product form.

$$\begin{aligned} y &= \frac{e^3x}{x^3} \\ &= e^{3x}\left(x^{-3}\right) \end{aligned}$$

Let $g(x) = e^{3x}$ and $h(x) = x^{-3}$.

Step 2
Determine $g'(x)$ and $h'(x)$.

$$\begin{aligned} g(x) &= e^{3x} \\ g'(x) &= 3e^{3x} \end{aligned}$$

and

$$\begin{aligned} h(x) &= x^{-3} \\ h'(x) &= -3x^{-4} \end{aligned}$$

Step 3
Apply the product law to determine the derivative of $y = \dfrac{e^3x}{x^3}$.

$$\begin{aligned} y &= g(x)h(x) \\ \frac{dy}{dx} &= g'(x)h(x) + g(x)h'(x) \\ \frac{dy}{dx} &= \left(3e^{3x}\right)\left(x^{-3}\right) + \left(e^{3x}\right)\left(-3x^{-4}\right) \\ \frac{dy}{dx} &= 3x^{-3}e^{3x} - 3x^{-4}e^{3x} \\ \frac{dy}{dx} &= 3x^{-4}e^{3x}(x - 1) \\ \frac{dy}{dx} &= \frac{3e^{3x}(x-1)}{x^4} \end{aligned}$$

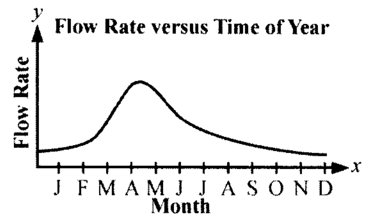
36. WR

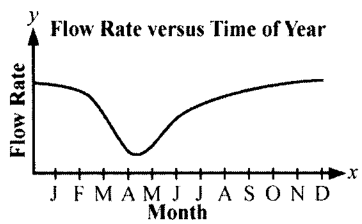
When finding the derivative of tu, apply the product law.

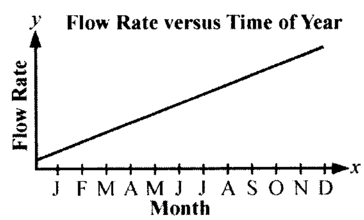
$$\begin{aligned} (tu)'(x) &= t'(x)u(x) + t(x)u'(x) \\ (tu)'(3) &= t'(3)u(3) + t(3)u'(3) \\ &= (5)(-2) + (4)(-1) \\ &= -10 - 4 \\ &= -14 \end{aligned}$$

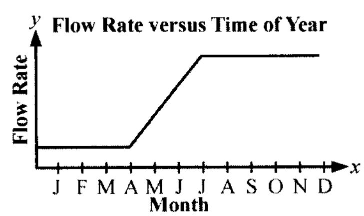
UNIT TEST — RATE OF CHANGE

1. Which of the following graphs **best** represents the flow rate of a glacier-fed river in Canada as a function of the month of the year?

A.


B.


C.


D.


Use the following information to answer the next question.

The given table shows the rate of change of some entity with respect to time for a particular situation.

Time	1	4	8	10
Rate of change with respect to time	2	16	256	1 024

2. Which of the following situations would **most likely** create the data in the given table?

A. A farmer recording population numbers when breeding rabbits

B. A passenger recording speeds as a vehicle accelerates to highway speeds

C. A pilot recording the air temperature as an airplane descends from 37 000 ft

D. A student recording the temperature on a thermometer when it is placed in a pot of boiling water

Use the following information to answer the next question.

Newton's law of cooling states that the rate of change of the temperature of an object is proportional to the difference between its own temperature and the temperature of its surroundings.

3. A thermometer that has been stored at room temperature is placed in a bucket of ice. The absolute value of the instantaneous rate of change in the thermometer's temperature reading will be

A. constant

B. greatest immediately after being inserted in the ice

C. greatest as it approaches the temperature of the ice

D. equal to the average rate of change over a given time interval

Use the following information to answer the next question.

For five years, the front porch of a home has been sinking a little bit more each year. The given table shows the year and corresponding sinking depth of the porch.

Year	Sinking Depth (cm)
2000	1.50
2001	3.10
2002	4.80
2003	6.70
2004	8.70
2005	10.90

4. Calculated to the nearest hundredth, what was the average rate of change in the depth over this period?

A. 1.44 cm/a **B.** 1.56 cm/a

C. 1.57 cm/a **D.** 1.88 cm/a

Use the following information to answer the next question.

The position, s, in metres, of a particle at a time, t, in seconds, is given by the equation $s = 2t^2 - 3t + 3$, where $t \geq 0$.

5. Which of the following pairs of points would provide the **best** approximation of the instantaneous rate of change when the particle's position is 8 m?

A. $t = 0.4$ s and $t = 0.6$ s

B. $t = 0.9$ s and $t = 1.1$ s

C. $t = 1.4$ s and $t = 1.6$ s

D. $t = 2.4$ s and $t = 2.6$ s

Use the following information to answer the next question.

The position, s, in metres, of an object after a time, t, in seconds, is defined by the function $s = t^2 + 2t - 8$.

Numerical Response

6. Calculated to the nearest tenth, the instantaneous rate of change of the object's position with respect to time when $t = 3.5$ is ________ m/s.

Use the following information to answer the next question.

The general term of a sequence is defined by the equation $t_n = 4 + \frac{1}{2n}$.

7. As n becomes infinitely large, t_n approaches

A. 0 **B.** 4

C. 4.5 **D.** infinity

Use the following information to answer the next question.

Jamie determined that the expression $\frac{f(5+h)-f(5)}{h}$ is 0 and concluded that the average rate of change of the function between $x = 5$ and $x = h$ is 0. Based on her findings, she made the following remarks:

I. The slope of the secant from $x = 5$ to $x = h$ is 0.
II. The slope of the tangent to the function at $x = 5$ is 0.
III. $f(5+h) = f(5)$
IV. The function has a maximum at $x = 5$.

8. Which of the given statements are certain?

A. I and II
B. I and III
C. II and IV
D. III and IV

Written Response

9. Provide an example of a function, *f*, for which the evaluation of the expression $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ would be unnecessary for determining the instantaneous rate of change of *f* at $x = a$.

Written Response

10. In simplified form, determine an expression equivalent to $\frac{f(-1+h)-f(-1)}{h}$ for the function $f(x) = 3x^2 - 1$.

Written Response

11. What expression can be used to approximate the instantaneous rate of change of a function or to calculate the average rate of change of a function?

12. Which of the following statements about a smooth and continuous curve is **true**?

A. To the immediate left of a local minimum, the slopes of the tangents to the curve are positive.

B. To the immediate left of a local maximum, the slopes of the tangents to the curve are negative.

C. To the immediate right of a local minimum, the slopes of the tangents to the curve are positive.

D. On either immediate side of a local maximum, the slopes of the tangents to the curve are negative.

13. For which of the following values of x is the instantaneous rate of change of the function $y = x^4 + 3x^3 + 6x^2 - 4x - 6$ equal to zero?

A. 0 only
B. 0.25 only
C. 0 and –2
D. 0.25 and –2

Use the following information to answer the next question.

A table of values for a continuous function and the slopes of its tangents is given.

x	$f(x)$	Slope of Tangent
−10	$-87.\bar{3}$	35
−9	−58	24
−8	$-38.\bar{6}$	15
−7	$-27.\bar{3}$	8
−6	−22	3
−5	$-20.\bar{6}$	0
−4	$-21.\bar{3}$	−1
−3	−22	0
−2	$-20.\bar{6}$	3
−1	$-15.\bar{3}$	8
	−4	15

Written Response

14. Assuming the slope of the tangent is zero only at the x-values indicated in the table, and its sign changes only at these values, identify the intervals or values for which the instantaneous rates of change of the given function are positive, negative, and zero.

15. If a polynomial function is quadratic, then the derivative of the function is

A. cubic　　**B.** linear

C. constant　　**D.** quadratic

16. Using the definition of a derivative, the value of $f'(5)$ for $f(x) = -3x^2 + 4x$ is determined from

A. $\lim\limits_{h \to 0} \dfrac{-3(5+h)^2 - 4(5+h) - \left(3(5)^2 - 4(5)\right)}{h}$

B. $\lim\limits_{h \to 0} \dfrac{-3(5+h)^2 + 4(5+h) - \left(3(5)^2 + 4(5)\right)}{h}$

C. $\lim\limits_{h \to 0} \dfrac{-3(5+h)^2 + 4(5+h) - \left(-3(5)^2 + 4(5)\right)}{h}$

D. $\lim\limits_{h \to 0} \dfrac{-3(5+h)^2 + 4(5+h) - \left(-3(5)^2 - 4(5)\right)}{h}$

Written Response

17. Use the definition of the derivative to determine $\dfrac{dy}{dx}$ for $y = x^3 - 4x + 9$.

18. If the derivative of a certain function is $f'(x) = 2\sin(x)$, the function is

A. $f(x) = 2\cos(x)$

B. $f(x) = \cos(2x)$

C. $f(x) = -2\cos(x)$

D. $f(x) = -\cos(2x)$

19. Which of the following tables of values corresponds to the instantaneous rates of change, $f'(x)$, of the function $f(x) = -\cos x$?

A.

x	$f'(x)$
0	0
$\frac{\pi}{4}$	0.7071
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	0.7071
π	0

B.

x	$f'(x)$
0	0
$\frac{\pi}{4}$	–0.7071
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	–0.7071
π	0

C.

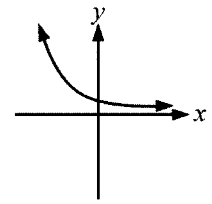
x	$f'(x)$
0	1
$\frac{\pi}{4}$	0.7071
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	–0.7071
π	–1

D.

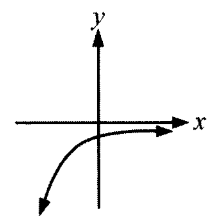
x	$f'(x)$
0	–1
$\frac{\pi}{4}$	–0.7071
$\frac{\pi}{2}$	0
$\frac{3\pi}{4}$	0.7071
π	1

20. Which of the following graphs could represent the graph of the derivative of the function $f(x) = \left(\frac{2}{5}\right)^{2x}$?

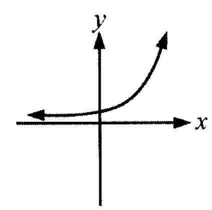
A.



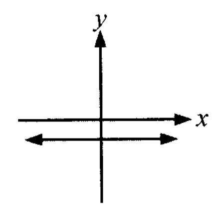
B.



C.



D.



21. For which of the following types of functions could the ratio between the derivative of the function and the function itself be constant?

A. Linear
B. Sinusoidal
C. Polynomial
D. Exponential

Numerical Response

22. Given the function $f(x) = e^x$, the value of $f'(2.5)$ to the nearest tenth is _________.

Numerical Response

23. Rounded to the nearest whole number, the value of x in the equation $\ln\left(e^{2x-1}\right) = 5$ is _________.

24. To the nearest thousandth, the slope of the tangent of the function $f(x)=\left(\frac{1}{3}\right)^x$ at $x = 3$ is

A. –0.041
B. 0.037
C. 0.041
D. 1.000

25. Which of the following equations uses the definition of a derivative to verify the power rule for finding the derivative of $f(x)=x^4$?

A. $f(x)=x^4$
B. $f'(x)=4x^3$
C. $f'(x)=\frac{(x+h)^4-x^4}{h}$
D. $f'(x)=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$

26. If $f(x)=3x^3$ and $g(x)=-2f(x)$, then the graph of $g'(x)$ is the graph of $f'(x)$ after a vertical stretch by a factor of

A. 2 about the x-axis
B. $\frac{1}{2}$ about the x-axis
C. 2 about the x-axis and a reflection in the x-axis
D. $\frac{1}{2}$ about the x-axis and a reflection in the x-axis

27. Which of the following functions have the same derivative?

A. $f(x)=3x^2$ and $f(x)=2x^3$
B. $f(x)=4x$ and $f(x)=x+4$
C. $f(x)=5x^2-12$ and $f(x)=5x^2+7$
D. $f(x)=x^4-5x+4$ and $f(x)=2x^2-5x+4$

28. At what point on the graph of the function $f(x)=x^3+2x^2-11x+15$ does the slope of the tangent to the function equal -12?

A. (1, 7)
B. (1, –4)
C. (–1, 27)
D. (–1, –12)

Numerical Response

29. The functions $f(x)=\frac{2}{3}x^3-\frac{7}{2}x^2+x-9$ and $g(x)=\frac{1}{3}x^3-\frac{3}{2}x^2-3x+5$ have identical instantaneous rates of change when the value of x, to the nearest whole number, is __________.

30. Which of the following expressions could be used to verify the product rule for finding the derivative of the function $f(x)=(2x-1)^3$?

A. $6x^3-1$
B. $8x^3-1$
C. $8x^3-12x^2+6x$
D. $8x^3-12x^2+6x-1$

31. If p, q, and r are non-zero differentiable functions in terms of x, then the derivative of $y=\frac{pq}{r}$ can be written as

A. $\frac{dy}{dx}=\frac{p'q'r-pqr'}{r^2}$
B. $\frac{dy}{dx}=\frac{pq'r+p'qr-pqr'}{r^2}$
C. $\frac{dy}{dx}=\frac{pqr'-pq'r-p'qr}{r^2}$
D. $\frac{dy}{dx}=\frac{p'qr+pq'r+pqr'}{r^2}$

Numerical Response

32. To the nearest hundredth, what is the value of x for which the tangent line of the function $f(x) = e^{3x-2}$ is equal to 50? ________

Numerical Response

33. Rounded to the nearest hundredth, the value of $|f'(\pi)|$ when $f(x) = \dfrac{\cos(x+\pi)}{4e^{2x^2-20}}$ is ________.

Written Response

34. If the coordinates are rounded to the nearest hundredth, at what point will the line tangent to the curve $y = 2xe^x$ be parallel to the line $y - x - 1 = 0$?

Written Response

35. Determine the derivative of the function $y = \sin(\sin x)$.

Written Response

36. Sketch the graphs of $f'(x) + g'(x)$ and $(f+g)'(x)$ given $f(x) = \frac{1}{3}x^3$ and $g(x) = -4x$.

Written Response

37. At what point on the graph of $f(x) = x^2 - 3$ is the line tangent to $f(x)$ perpendicular to the line $2y - x + 1 = 0$?

Answers and Solutions — Unit Test

1. A	9. WR	17. WR	25. D	33. 4.08
2. A	10. WR	18. C	26. C	34. WR
3. B	11. WR	19. A	27. C	35. WR
4. D	12. C	20. B	28. C	36. WR
5. D	13. B	21. D	29. 2	37. WR
6. 9.0	14. WR	22. 12.2	30. D	
7. B	15. B	23. 3	31. B	
8. B	16. C	24. A	32. 1.60	

1. A

The flow rate from January to March is very low, since most of the glacier is frozen. The flow rate from March to June increases rapidly as the warming temperatures melt the glacier. From June to September, the flow rate gradually decreases as the volume of the glacier decreases. Finally, from September to December, the flow rate is very low as the glacier becomes frozen.

2. A

The population of rabbits breeding over time is the only given situation that increases exponentially. This would mean that the rate of change increases rapidly, as illustrated in the given table.

3. B

The absolute value of the instantaneous rate of change in the thermometer's temperature just after insertion is very large since the difference between its temperature and the surrounding temperature of the ice is large. The absolute value of the instantaneous rate of change becomes less and less as the thermometer's temperature approaches the temperature of the ice.

4. D

The average rate of change over the five-year period is the slope, m, between the two points (0, 1.50) and (5, 10.90).

$$m = \frac{10.90 - 1.50}{5 - 0}$$
$$= \frac{9.40}{5}$$
$$= 1.88$$

The average rate of change in the depth is 1.88 cm/a.

5. D

Step 1
Determine the value(s) of t when the particle's position is 8 m.

$$s = 2t^2 - 3t + 3$$
$$8 = 2t^2 - 3t + 3$$
$$0 = 2t^2 - 3t - 5$$
$$t = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-5)}}{2(2)}$$
$$t = \frac{3 \pm \sqrt{49}}{4}$$
$$t = \frac{3 \pm 7}{4}$$
$$t = -1 \text{ or } 2.5$$

The time when the particle's position is 8 m is 2.5 s since $t = -1$ is extraneous ($t \geq 0$).

Step 2
Determine the pair of points that will best approximate the instantaneous rate of change at $t = 2.5$ s.

Of the given alternatives, $t = 2.4$ s and $t = 2.6$ s are the best points for approximating the instantaneous rate of change at $t = 2.5$ s because they are closest to $t = 2.5$ s and are located on either side of this point.

6. 9.0

Use technology or a table of values to determine the instantaneous rate of change at $t = 3.5$ by finding the slopes of secant lines that pass through the point at $t = 3.5$ and points with t-values close by, such as $t = 3.49$, 3.51, 3.499, and 3.501. Note that the point at $t = 3.5$ is $\left(3.5, (3.5)^2 + 2(3.5) - 8\right) = (3.5, 11.25)$.

The solution can be calculated by following the steps below with a TI-83 graphing calculator or a similar calculator.

Step 1
Clear the table entries. Press [2nd] [0], scroll down to ClrTable, and press [ENTER] twice.

Step 2

Press [Y=], and enter $Y_1 = X^2 + 2X - 8$.

Step 3

Enter a second function that will give the slopes of the secant lines for the various t-values of the second point.

The equation that will be entered is $Y_2 = (Y_1 - 11.25)/(X - 3.5)$. To enter this equation, place the cursor into $Y_2 =$. Press [(] and then [VARS]. Scroll right to access the Y-VARS Menu. Select 1:Function…, and press [ENTER] to bring up Y_1. Then, enter the rest of the equation.

Step 4

Press [2nd] [WINDOW], and highlight Indpnt: ASK, Depend: AUTO.

Step 5

Press [2nd] [GRAPH], and type the t-values 3.49, 3.51, 3.499, and 3.501 into the first column. (Press [ENTER] after each entry.) You should see the screen shot shown.

X	Y1	Y2
3.49	11.16	8.99
3.51	11.34	9.01
3.499	11.241	8.999
3.501	11.259	9.001

X=

From the values in Y_2 (the slopes of secants), the instantaneous rate of change of the object's position at $t = 3.5$ s, to the nearest tenth, is 9.0 m/s.

7. B

Find the limit of t_n as n approaches infinity.

$$\lim_{n \to \infty} 4 + \frac{1}{2n}$$
$$= \lim_{n \to \infty} \frac{8n+1}{2n}$$
$$= \lim_{n \to \infty} \frac{\frac{8n}{n} + \frac{1}{n}}{\frac{2n}{n}}$$
$$= \lim_{n \to \infty} \frac{8 + \frac{1}{n}}{2}$$
$$= \frac{8+0}{2}$$
$$= 4$$

8. B

For $\frac{f(5+h) - f(5)}{h}$ to equal 0, $f(5+h) - f(5) = 0$. Therefore, $f(5+h) = f(5)$ is certain. Also, the expression $\frac{f(5+h) - f(5)}{h}$ describes the slope of the secant from $x = 5$ to $x = h$, which is 0. Statements II and IV apply to the expression $\lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$ and therefore are not certain. Statements I and III are certain.

9. WR

A constant function $y = c$ is a horizontal line where the instantaneous rate of change is zero for all values of x. For such a function, it would be unnecessary to evaluate the expression $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ at $x = a$.

10. WR

Simplify $\frac{f(-1+h) - f(-1)}{h}$ for $f(x) = 3x^2 - 1$.

$$\frac{f(-1+h) - f(-1)}{h}$$
$$= \frac{\left(3(-1+h)^2 - 1\right) - \left(3(-1)^2 - 1\right)}{h}$$
$$= \frac{\left(3\left(1 - 2h + h^2\right) - 1\right) - (3 - 1)}{h}$$
$$= \frac{3 - 6h + 3h^2 - 1 - 2}{h}$$
$$= \frac{-6 + 3h^2}{h}$$
$$= \frac{h(-6 + 3h)}{h}$$
$$= -6 + 3h$$

11. WR

The expression $\frac{f(a+h) - f(a)}{h}$ can be used to approximate the instantaneous rate of change at $x = a$ for a function, f, or to calculate the average rate of change from $x = a + h$ to $x = a$.

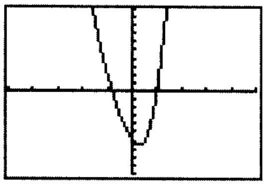
12. C

At a local maximum, the slopes of the tangents to a smooth and continuous curve change from positive (to its left) to negative (to its right).

At a local minimum, the slopes of the tangents to a smooth and continuous curve change from negative (to its left) to positive (to its right).

13. B

Using a graphing calculator, graph the function using the window settings x: $[-5, 5, 1]$ and y: $[-10, 10, 1]$.



By observing the graph, it is evident that there is a minimum point at about $x = 0.25$. The slope of the tangent at this point (instantaneous rates of change) would equal zero.

14. WR

There are positive instantaneous rates of change on the intervals $-10 \leq x < -5$ and $-3 < x \leq 0$ because the slopes of tangents are all positive.

There are negative instantaneous rates of change on the interval $-5 < x < -3$ because the slopes of the tangents are all negative.

The instantaneous rates of change are zero at $x = -5$ and at $x = -3$.

15. B

The derivative of a polynomial function of degree n has a degree of $n - 1$. Therefore, the derivative of a quadratic function ($n = 2$) is linear ($n = 1$).

16. C

The derivative of a function, $f(x)$, is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Substitute 5 into $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ to determine $f'(5)$.

$f'(5)$

$$= \lim_{h \to 0} \frac{f(5+h) - f(5)}{h}$$

$$= \lim_{h \to 0} \frac{-3(5+h)^2 + 4(5+h) - \left(-3(5)^2 + 4(5)\right)}{h}$$

17. WR

The derivative of a function, y, is as follows:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Use the definition of a derivative to solve for $\frac{dy}{dx}$ for the function $y = x^3 - 4x + 9$.

$\frac{dy}{dx}$

$$= \lim_{h \to 0} \frac{\left((x+h)^3 - 4(x+h) + 9\right) - \left(x^3 - 4x + 9\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left((x+h)(x^2 + 2xh + h^2) - 4x - 4h + 9\right) - \left(x^3 - 4x + 9\right)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3xh^2 + 3x^2h + h^3 - 4x - 4h + 9 - x^3 + 4x - 9}{h}$$

$$= \lim_{h \to 0} \frac{3xh^2 + 3x^2h + h^3 - 4h}{h}$$

$$= \lim_{h \to 0} \frac{h\left(3xh + 3x^2 + h^2 - 4\right)}{h}$$

$$= \lim_{h \to 0} 3xh + 3x^2 + h^2 - 4$$

$$= 3x^2 - 4$$

18. C

The derivative of $f(x) = \cos(x)$ is $f'(x) = -\sin(x)$. Therefore, the derivative of $f(x) = -2\cos(x)$ is $f'(x) = 2\sin(x)$.

This result can also be verified by using a TI-83 or similar calculator. To do so, follow the steps below.

Step 1

Press [Y =], enter $-2\cos(x)$ in Y_1, and press [ENTER].

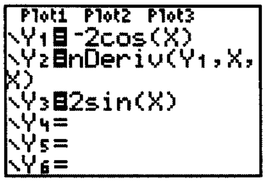
Step 2

Press [MATH], and select 8:nDeriv(. Press [VARS], and scroll right to select Y-VARS. Select 1:Function... and then 1:Y_1. Press [,] to insert a comma, and enter X, X). Press [ENTER].

This procedure will enter the derivative of $f(x) = -2\cos(x)$ into Y_2.

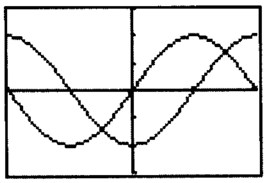
Step 3
Enter 2sin (x) in Y_3.

The three entries should appear as shown.



Next, press GRAPH. (Make sure that an appropriate window setting has been selected such as $x:\left[-\pi, \pi, \frac{\pi}{2}\right]$ and $y:[-3, 3, 1]$.)

Graphs Y_2 and Y_3 should be identical.



19. A

Clear all entries in the table in your calculator. Press 2nd 0 (CATALOG), then scroll to ClrTable and press ENTER twice.

Next, press Y = and enter the function $y = -\cos(x)$ and its derivative.
$Y_1 = -\cos(X)$
$Y_2 = \text{nDeriv}(Y_1, X, X)$

Then press 2nd GRAPH and enter the values $x = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi$.

Compare the values in the table to identify the correct table.

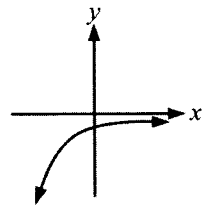
This table corresponds to the derivative of the function $f(x) = -\cos x$.

x	$f'(x)$
0	0
$\frac{\pi}{4}$	0.7071
$\frac{\pi}{2}$	1
$\frac{3\pi}{4}$	0.7071
π	0

20. B

Press Y = on a graphing calculator, and enter the function and its derivative with a window setting of $x:[-1.5, 1.5, 1]$, $y:[-10, 10, 1]$.
$Y_1 = (2/5)\wedge(2X)$
$Y_2 = \text{nDeriv}(Y_1, X, X)$

Press GRAPH. Notice that the graph of the derivative of $f(x)$ looks like the given graph.



21. D

When an exponential function $f(x)$ is divided by its derivative function $f'(x)$, the quotient is
$\frac{f'(x)}{f(x)} = k$, where k is a constant.

22. 12.2

The derivative of $f(x) = e^x$ is $f'(x) = e^x$.
Solve for $f'(2.5)$.
$f'(2.5) = e^{2.5}$
$= 12.182\ 493\ 96$

Rounded to the nearest tenth, $f'(2.5)$ is 12.2.

23. 3

Step 1
Since $\ln(e^x) = x$, then $\ln(e^{2x-1}) = 2x - 1$.
Substitute $2x - 1$ for $\ln(e^{2x-1})$ in the equation $\ln(e^{2x-1}) = 5$.
$\ln(e^{2x-1}) = 5$
$2x - 1 = 5$

Step 2
Solve for x.
$2x - 1 = 5$
$2x = 6$
$x = 3$

24. A

The slope of the tangent for a function $y = f(x)$ is the derivative of $f(x)$ at any value of x.
The derivative of $f(x) = a^x$ is $f'(x) = a^x \ln(a)$.

Calculate $f'(3)$ for the function $f(x)=\left(\frac{1}{3}\right)^x$.

$f'(3)$
$=\left(\frac{1}{3}\right)^x\ln\left(\frac{1}{3}\right)$
$=\left(\frac{1}{3}\right)^3\ln\left(\frac{1}{3}\right)$
$\approx(0.037\ 037\ 037)(-1.098\ 612\ 289)$
≈ -0.041

25. D

According to the power rule, $f'(x)=nx^{n-1}$. Therefore, the derivative of $f(x)=x^4$ is $f'(x)=4x^3$.

The definition of a derivative is $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$. Calculate the derivative of $f(x)=x^4$.

$f'(x)$
$=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$
$=\lim_{h\to 0}\frac{\left(x^4+4x^3h+6x^2h^2+4xh^3+h^4\right)-x^4}{h}$
$=\lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h}$
$=\lim_{h\to 0}4x^3+6x^2h+4xh^2+h^3$
$=4x^3$

The result matches the result from the power rule.

Therefore, the equation $f'(x)=\lim_{h\to 0}\frac{(x+h)^4-x^4}{h}$ uses the definition of a derivative to verify the power rule.

26. C

The constant multiple rule states that the derivative of $g(x)=kf(x)$, $k\in\mathbb{R}$, is $g'(x)=kf'(x)$.

If $g(x)=-2f(x)$, then $g'(x)=-2f'(x)$. Therefore, $g'(x)$ is the graph of $f'(x)$ multiplied by a factor of -2, which causes a vertical stretch by a factor of 2 about the x-axis and a reflection in the x-axis.

27. C

Step 1

Take the derivative of $f(x)=3x^2$ and $f(x)=2x^3$.

$f(x)=3x^2$
$f'(x)=6x$
$f(x)=2x^3$
$f'(x)=6x^2$

The derivatives are not the same.

Step 2

Take the derivative of $f(x)=4x$ and $f(x)=x+4$.

$f(x)=4x$
$f'(x)=4$
$f(x)=x+4$
$f'(x)=1$

The derivatives are not the same.

Step 3

Take the derivative of $f(x)=5x^2-12$ and $f(x)=5x^2+7$.

$f(x)=5x^2-12$
$f'(x)=10x$
$f(x)=5x^2+7$
$f'(x)=10x$

The derivatives are the same.

Step 4

Take the derivative of $f(x)=x^4-5x+4$ and $f(x)=2x^2-5x+4$.

$f(x)=x^4-5x+4$
$f'(x)=4x^3-5$
$f(x)=2x^2-5x+4$
$f'(x)=4x-5$

The derivatives are not the same.

28. C

Step 1

Take the derivative of the function $f(x)=x^3+2x^2-11x+15$.

$f(x)=x^3+2x^2-11x+15$
$f'(x)=3x^2+4x-11$

Step 2

Since the slope of the tangent is -12, $f'(x)=-12$. Substitute -12 for $f'(x)$ in the equation $f'(x)=3x^2+4x-11$.

$f'(x)=3x^2+4x-11$
$-12=3x^2+4x-11$
$0=3x^2+4x+1$

Step 3
Factor the equation, and solve for x.

$0 = 3x^2 + 4x + 1$
$0 = 3x^2 + 3x + x + 1$
$0 = 3x(x + 1) + (x + 1)$
$0 = (3x + 1)(x + 1)$

Therefore, the x-values are $x = -1$ and $x = -\frac{1}{3}$.

The only viable option is $x = -1$ because none of the given points have an x-value of $-\frac{1}{3}$.

Step 4
Substitute -1 for x in the function $f(x) = x^3 + 2x^2 - 11x + 15$, and solve for $f(-1)$.

$f(x) = x^3 + 2x^2 - 11x + 15$
$f(-1) = (-1)^3 + 2(-1)^2 - 11(-1) + 15$
$f(-1) = -1 + 2 + 11 + 15$
$f(-1) = 27$

Therefore, the point on the graph of $f(x) = x^3 + 2x^2 - 11x + 15$ where the slope of the tangent is -12 is $(-1, 27)$.

29. 2

Step 1
Take the derivative of the function $f(x) = \frac{2}{3}x^3 - \frac{7}{2}x^2 + x - 9$.

$f(x) = \frac{2}{3}x^3 - \frac{7}{2}x^2 + x - 9$
$f'(x) = 2x^2 - 7x + 1$

Step 2
Take the derivative of the function $g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 3x + 5$.

$g(x) = \frac{1}{3}x^3 - \frac{3}{2}x^2 - 3x + 5$
$g'(x) = x^2 - 3x - 3$

Step 3
Since the instantaneous rate of change is found by taking the derivative of a function, for the two functions to have identical instantaneous rates of change, $f'(x) = g'(x)$. Equate the derivatives of the two functions, and bring all the terms to one side.

$f'(x) = g'(x)$
$2x^2 - 7x + 1 = x^2 - 3x - 3$
$x^2 - 4x + 4 = 0$

Step 4
Factor the equation, and solve for x.

$x^2 - 4x + 4 = 0$
$(x - 2)(x - 2) = 0$
$(x - 2)^2 = 0$
$x = 2$

Therefore, the functions $f(x)$ and $g(x)$ have identical instantaneous rates of change at $x = 2$.

30. D

Simplify the function $f(x) = (2x - 1)^3$.

$$\begin{aligned} f(x) &= (2x - 1)^3 \\ &= (2x - 1)(2x - 1)^2 \\ &= (2x - 1)(4x^2 - 4x + 1) \\ &= 8x^3 - 12x^2 + 6x - 1 \end{aligned}$$

The function $f(x) = 8x^3 - 12x^2 + 6x - 1$ can be used to verify the product rule for finding the derivative of the function $f(x) = (2x - 1)^3$.

31. B

Step 1
Rewrite the function $y = \frac{pq}{r}$ in product form.

$$\begin{aligned} y &= \frac{pq}{r} \\ &= pqr^{-1} \end{aligned}$$

Step 2
Substitute f for pq, and apply the product rule to the function.

$y = pqr^{-1}$
$y = fr^{-1}$
$\frac{dy}{dx} = f'r^{-1} + f(-1r^{-2}(r'))$
$\frac{dy}{dx} = f'r^{-1} - fr'r^{-2}$

Step 3
Determine f' by applying the product rule to the function $f = pq$.

$f = pq$
$f' = pq' + p'q$

Step 4
Substitute $pq' + p'q$ for f' into the derivative $\frac{dy}{dx} = f'r^{-1} - fr'r^{-2}$, and simplify.

$$\begin{aligned} \frac{dy}{dx} &= f'r^{-1} - fr'r^{-2} \\ &= (pq' + p'q)r^{-1} - (pq)r'r^{-2} \\ &= pq'r^{-1} + p'qr^{-1} - pqr'r^{-2} \\ &= \frac{pq'r + p'qr - pqr'}{r^2} \end{aligned}$$

32. 1.60

Step 1

Take the derivative of the function $f(x) = e^{3x-2}$. Apply the chain rule.

$f(x) = e^{3x-2}$

$f'(x) = e^{3x-2}\left(\frac{d}{dx}(3x-2)\right)$

$= e^{3x-2}(3)$

$= 3e^{3x-2}$

Step 2

Since $f'(x)$ is equal to the tangent line, substitute 50 for $f'(x)$ in the derivative function $f'(x) = 3e^{3x-2}$, and solve for x.

$f'(x) = 3e^{3x-2}$

$50 = 3e^{3x-2}$

$\frac{50}{3} = e^{3x-2}$

$\ln\left(\frac{50}{3}\right) = \ln e^{3x-2}$

$\ln\left(\frac{50}{3}\right) = 3x-2$

$\ln\left(\frac{50}{3}\right) + 2 = 3x$

$\frac{\ln\left(\frac{50}{3}\right)+2}{3} = x$

$1.604\,4702\ldots = x$

$1.60 \approx x$

33. 4.08

Step 1

Change the function from quotient form to product form.

$f(x) = \frac{\cos(x+\pi)}{4e^{2x^2-20}}$

$= (\cos(x+\pi))\left(\frac{1}{4}e^{-(2x^2-20)}\right)$

$= (\cos(x+\pi))\left(\frac{1}{4}e^{-2x^2+20}\right)$

Step 2

Let $g(x) = \cos(x+\pi)$ and $h(x) = \frac{1}{4}e^{-2x^2+20}$.

Determine $g'(x)$ and $h'(x)$ by applying the chain rule.

$g(x) = \cos(x+\pi)$

$g'(x) = -\sin(x+\pi)\left(\frac{d}{dx}(x+\pi)\right)$

$g'(x) = -\sin(x+\pi)(1)$

$g'(x) = -\sin(x+\pi)$

$h(x) = \frac{1}{4}e^{-2x^2+20}$

$h'(x) = \frac{1}{4}e^{-2x^2+20}\left(\frac{d}{dx}(-2x^2+20)\right)$

$h'(x) = \frac{1}{4}e^{-2x^2+20}(-4x)$

$h'(x) = -xe^{-2x^2+20}$

Step 3

Determine the derivative of $f(x)$ by applying the product rule.

$f(x) = g(x)h(x)$

$f'(x) = g'(x)h(x) + g(x)h'(x)$

$f'(x) = \left((-\sin(x+\pi))\left(\frac{1}{4}e^{-2x^2+20}\right) + (\cos(x+\pi))\left(-xe^{-2x^2+20}\right)\right)$

Step 4

Substitute π for x in the derivative function $f'(x) = \left((-\sin(x+\pi))\left(\frac{1}{4}e^{-2x^2+20}\right) + (\cos(x+\pi))\left(-xe^{-2x^2+20}\right)\right)$, and solve for $f'(\pi)$.

$f'(x) = \left((-\sin(x+\pi))\left(\frac{1}{4}e^{-2x^2+20}\right) + (\cos(x+\pi))\left(-xe^{-2x^2+20}\right)\right)$

$f'(\pi) = \left((-\sin(\pi+\pi))\left(\frac{1}{4}e^{-2\pi^2+20}\right) + (\cos(\pi+\pi))\left(-xe^{-2\pi^2+20}\right)\right)$

$f'(\pi) = \left((-\sin(2\pi))\left(\frac{1}{4}e^{-2\pi^2+20}\right) + (\cos(2\pi))\left(-\pi e^{-2\pi^2+20}\right)\right)$

$f'(\pi) = (0)\left(\frac{1}{4}e^{0.260\,79\ldots}\right) + (1)\left(-\pi e^{0.260\,79\ldots}\right)$

$f'(\pi) = -\pi e^{0.260\,79\ldots}$

$f'(\pi) \approx -4.077\,65\ldots$

Rounded to the nearest hundredth, $|f'(\pi)| = 4.08$.

34. WR

Step 1
Rearrange the equation $y - x - 1 = 0$ in the form $y = mx + b$.

$y - x - 1 = 0$
$y = x + 1$

Notice that the line has a slope of 1. For the tangent of the curve $y = 2xe^x$ to be parallel to the line, the derivative of $y = 2xe^x$ must be equal to 1.

Step 2
Apply the product rule to determine the derivative of $y = 2xe^x$.

$y = 2xe^x$
$y' = (2x)(e^x) + (2)(e^x)$
$y' = 2xe^x + 2e^x$
$y' = 2e^x(x+1)$

Step 3
Substitute 1 for y' into the derivative $y' = 2e^x(x+1)$, and bring all the terms to one side.

$y' = 2e^x(x+1)$
$1 = 2e^x(x+1)$
$0 = 2e^x(x+1) - 1$

Step 4
Determine the value of x.
Using a graphing calculator, graph the equation $y = 2e^x(x+1) - 1$ and determine the zero. On a TI-83 graphing calculator, the key strokes to find the zeros of a function are as follows: press [2nd] [TRACE] and select 2:zero.
The graphing calculator will give the value of −0.314 9231.
Therefore, $x = -0.314\,9231$ when the tangent line of the curve $y = 2xe^x$ is parallel to the line $y - x - 1 = 0$.

Step 5
Substitute −0.314 9231 for x into the equation $y = 2xe^x$, and solve for y.

$y = 2xe^x$
$y = 2(-0.3149\ldots)e^{-0.3149\ldots}$
$y = -0.459\,66\ldots$

Therefore, the coordinates when the tangent line of the curve $y = 2xe^x$ is parallel to the line $y - x - 1 = 0$ is (−0.31, −0.46), rounded to the nearest hundredth.

35. WR

Use the chain rule to find the derivative of $y = \sin(\sin x)$.

$y = \sin(\sin x)$
$\frac{dy}{dx} = \cos(\sin x)\left(\frac{d}{dx}\sin x\right)$
$= \cos(\sin x)(\cos x)$
$= \cos^2 x(\sin x)$

36. WR

Step 1
Determine $f'(x)$ and $g'(x)$.

$f(x) = \frac{1}{3}x^3$
$f'(x) = x^2$
$g(x) = -4x$
$g'(x) = -4$

Step 2
Determine $f'(x) + g'(x)$.

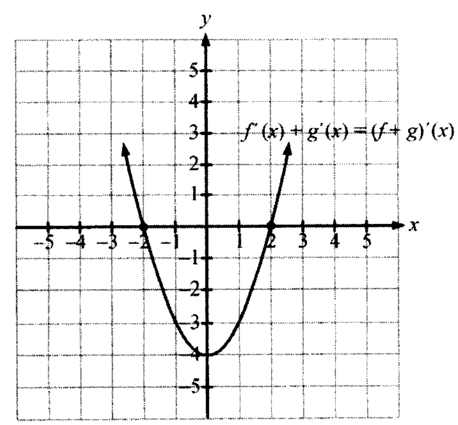
$f'(x) + g'(x) = x^2 - 4$

Step 3
Determine $(f+g)'(x)$.

$(f+g)(x) = f(x) + g(x)$
$= \frac{1}{3}x^3 - 4x$
$(f+g)'(x) = x^2 - 4$

Step 4
Sketch the graphs of $f'(x) + g'(x)$ and $(f+g)'(x)$.
The graphs of $f'(x) + g'(x)$ and $(f+g)'(x)$ are identical.

$f'(x) + g'(x) = x^2 - 4$
$(f+g)'(x) = x^2 - 4$



37. WR

Step 1

Rearrange the equation $2y - x + 1 = 0$ into the form $y = mx + b$.

$$2y - x + 1 = 0$$
$$2y = x - 1$$
$$y = \frac{1}{2}x - \frac{1}{2}$$

Step 2

Determine the slope of a line perpendicular to the line $y = \frac{1}{2}x - \frac{1}{2}$.

The slope of the line is $\frac{1}{2}$. To determine the slope of a perpendicular line, take the negative reciprocal of $\frac{1}{2}$. Therefore, the slope of a perpendicular line is as follows:

$$m_{\perp} = \frac{1}{-\frac{1}{2}}$$
$$= -2$$

Step 3

Determine the derivative of the function $f(x) = x^2 - 3$.

$$f(x) = x^2 - 3$$
$$f'(x) = 2x$$

The derivative of a function gives the slope of the tangent, so for the tangent line to be perpendicular to the line $2y - x + 1 = 0$, $f'(x) = -2$.

Step 4

Substitute -2 for $f'(x)$ into the equation $f'(x) = 2x$, and solve for x.

$$f'(x) = 2x$$
$$-2 = 2x$$
$$-1 = x$$

Step 5

Substitute -1 for x into the function $f(x) = x^2 - 3$, and solve for $f(-1)$.

$$f(x) = x^2 - 3$$
$$f(-1) = (-1)^2 - 3$$
$$f(-1) = 1 - 3$$
$$f(-1) = -2$$

Therefore, the point on the graph of $f(x) = x^2 - 3$ where its tangent line is perpendicular to the line $2y - x + 1 = 0$ is $(-1, -2)$.

NOTES

Derivatives and their Applications

DERIVATIVES AND THEIR APPLICATIONS

Table of Correlations

	Outcome	Practice Questions	Unit Test Questions	Practice Test 1	Practice Test 2
DER1.0	Connecting Graphs and Equations of Functions and Their Derivatives				
DER1.1	*sketch the graph of a derivative function, given the graph of a function that is continuous over an interval, and recognize points of inflection of the given function (i.e., points at which the concavity changes)*	1, 2, 3, 4	1, 2a, 2b		17
DER1.2	*recognize the second derivative as the rate of change of the rate of change (i.e., the rate of change of the slope of the tangent), and sketch the graphs of the first and second derivatives, given the graph of a smooth function*	5	3, 4, 5, 6	17	18
DER1.3	*determine algebraically the equation of the second derivative $f''(x)$ of a polynomial or simple rational function $f(x)$, and make connections, through investigation using technology, between the key features of the graph of the function (e.g., increasing/decreasing intervals, local maxima and minima, points of inflection, intervals of concavity) and corresponding features of the graphs of its first and second derivatives (e.g., for an increasing interval of the function, the first derivative is positive; for a point of inflection of the function, the slopes of tangents change their behaviour from increasing to decreasing or from decreasing to increasing, the first derivative has a maximum or minimum, and the second derivative is zero)*	6, 7, 8	7, 8, 9	18, 19, 20	19, 20, 21
DER1.4	*describe key features of a polynomial function, given information about its first and/or second derivatives (e.g., the graph of a derivative, the sign of a derivative over specific intervals, the x-intercepts of a derivative), sketch two or more possible graphs of the function that are consistent with the given information, and explain why an infinite number of graphs is possible*	9, 10, 11	10, 11	21, 22	22, 45
DER1.5	*sketch the graph of a polynomial function, given its equation, by using a variety of strategies (e.g., using the sign of the first derivative; using the sign of the second derivative; identifying even or odd functions) to determine its key features (e.g., increasing/ decreasing intervals, intercepts, local maxima and minima, points of inflection, intervals of concavity), and verify using technology*	12, 13, 14	12, 13, 14, 15	23, 24	23
DER2.0	Solving Problems Using Mathematical Models and Derivatives				
DER2.1	*make connections between the concept of motion (i.e., displacement, velocity, acceleration) and the concept of the derivative in a variety of ways (e.g., verbally, numerically, graphically, algebraically)*	15, 16, 17	16, 17, 18	25	24
DER2.2	*make connections between the graphical or algebraic representations of derivatives and real-world applications (e.g., population and rates of population change, prices and inflation rates, volume and rates of flow, height and growth rates)*	18, 19, 20	19, 20	26	25
DER2.3	*solve problems, using the derivative, that involve instantaneous rates of change, including problems arising from real-world applications (e.g., population growth, radioactive decay, temperature changes, hours of day-light, heights of tides), given the equation of a function**	21, 22, 23	21, 22a, 22b, 23, 24, 25	27	26

DER2.4	*solve optimization problems involving polynomial, simple rational, and exponential functions drawn from a variety of applications, including those arising from real-world situations*	24, 25, 26, 27	26, 27, 28	28, 29	27, 46
DER2.5	*solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate the results*	28, 29, 30	29, 30, 31, 32	30	28

DER1.1 sketch the graph of a derivative function, given the graph of a function that is continuous over an interval, and recognize points of inflection of the given function (i.e., points at which the concavity changes)

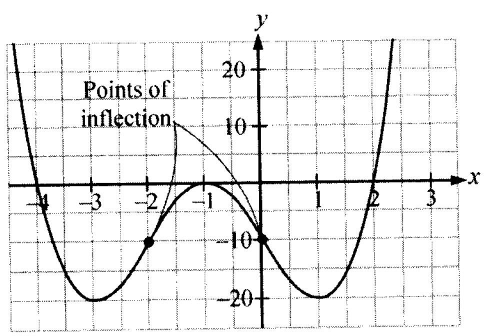
DER1.2 recognize the second derivative as the rate of change of the rate of change (i.e., the rate of change of the slope of the tangent), and sketch the graphs of the first and second derivatives, given the graph of a smooth function

Graphing the First and Second Derivative Function

When the graph or type of a polynomial function is given, the first derivative function over a continuous interval can be sketched by approximating the slope of the graph, which is the slope of the tangent line, at various x-values.

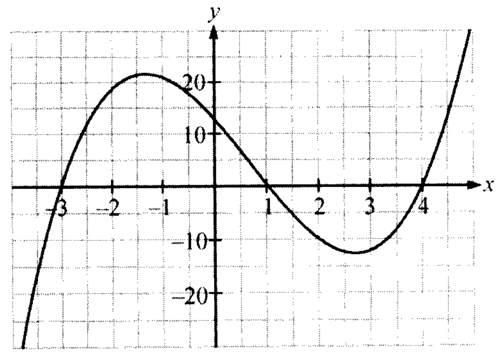
A point on a graph of a function where the **concavity** of the graph changes from concave up to concave down, or vice versa, is called a **point of inflection**.

The points of inflection on the given graph are located at approximately $x = -2$ and $x = 0$.



In the same way that the graph of the first derivative function of a given graph is sketched, the graph of the second derivative function can be drawn by approximating the slope of the graph of the first derivative function at various x-values.

Example



The graph of a cubic function is shown.

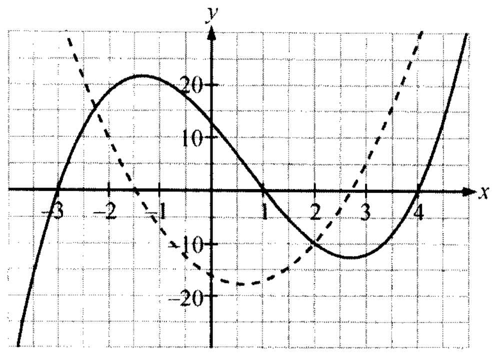
On the same set of axes, sketch the graphs of the first and second derivatives of the given cubic function.

Solution

Step 1
Analyze the slope of the graph from left to right using approximate x-values.

- When $x < -1.4$, the slope is positive and decreasing.
- When $x = -1.4$, the slope is 0.
- When $-1.4 < x < 0.7$, the slope is negative and decreasing.
- When $0.7 < x < 2.7$, the slope is negative and increasing.
- When $x = 2.7$, the slope is 0.
- When $x > 2.7$, the slope is positive and increasing.

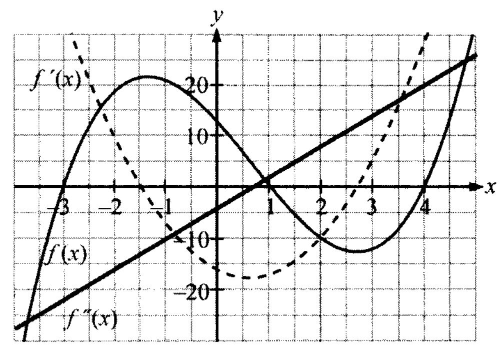
Step 2
Graph the first derivative.
Graph the approximate slope values and the corresponding x-values on the same set of axes as the original function. Draw a smooth curve through these points to produce the curve shown here by the dotted line. Since the original graph is a cubic function, the first derivative is quadratic; thus, its graph is a parabola.



Step 3
Analyze the slope of the graph of the first derivative using approximate x-values.

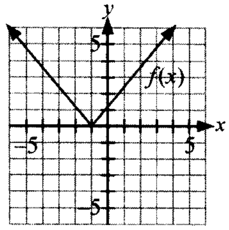
- When $x = -3$, the slope is negative, such as -22.
- When $x < 0.7$ the slope is negative and increasing.
- When $x > 0.7$ the slope is positive and increasing.
- When $x = 4$, the slope is a positive number, such as $+20$.

Step 4
Graph the second derivative.
Notice that as the x-values of the graph increase, the slope of the graph increases. Therefore, because the graph of the first derivative is a parabola, the second derivative is a linear function.
Two points are required to graph a linear function. Notice that the slope of the first derivative is 0 when x is approximately 0.7, and the slope is approximately 20 when x is 4. Graph the points (0.7, 0) and (4, 20), and draw a straight line through these points to produce the graph shown here by the thick solid line. This graph is the second derivative of the original function.



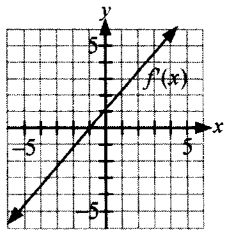
Use the following information to answer the next question.

The graph of $f(x)$ is shown.

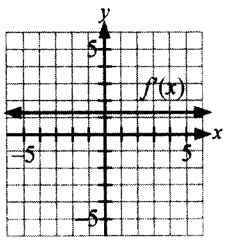


1. Which of the following graphs **best** represents the graph of $f'(x)$?

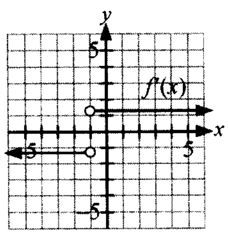
A.



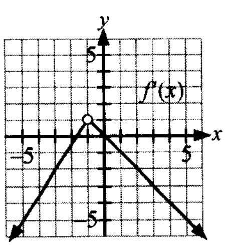
B.



C.

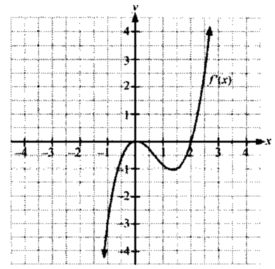


D.



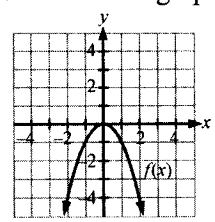
Use the following information to answer the next question.

The graph of the derivative of $f(x)$ is shown.

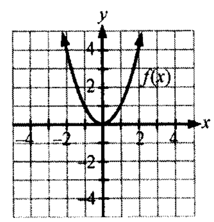


2. Which of the following graphs **best** approximates the graph of $f(x)$?

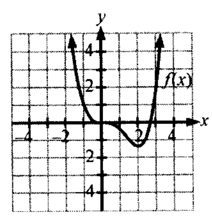
A.



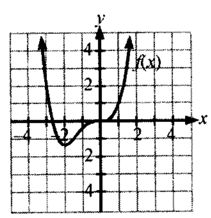
B.



C.

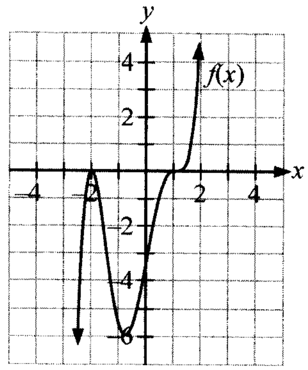


D.



Use the following information to answer the next question.

The graph of $f(x)$ is shown.



Written Response

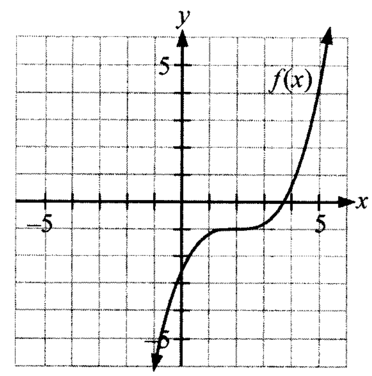
3. Given the graph of $f(x)$, sketch the graph of $f'(x)$.

Written Response

4. A function $f(x)$ is not differentiable at $x = a$. If the graph of $f(x)$ were shown, what might be two of the reasons that are evident on the graph of $f(x)$ for this lack of differentiability at $x = a$?

Use the following information to answer the next question.

The graph of $y = f(x)$ is shown.



5. For what approximate value of x is the instantaneous rate of change of $f(x)$ equal to 0?

A. 0 **B.** 0.7

C. 2.0 **D.** 3.3

DER1.3 determine algebraically the equation of the second derivative f''(x) of a polynomial or simple rational function f(x), and make connections, through investigation using technology, between the key features of the graph of the function (e.g., increasing/decreasing intervals, local maxima and minima, points of inflection, intervals of concavity) and corresponding features of the graphs of its first and second derivatives (e.g., for an increasing interval of the function, the first derivative is positive; for a point of inflection of the function, the slopes of tangents change their behaviour from increasing to decreasing or from decreasing to increasing, the first derivative has a maximum or minimum, and the second derivative is zero)

DER1.4 describe key features of a polynomial function, given information about its first and/or second derivatives (e.g., the graph of a derivative, the sign of a derivative over specific intervals, the x-intercepts of a derivative), sketch two or more possible graphs of the function that are consistent with the given information, and explain why an infinite number of graphs is possible

DER1.5 sketch the graph of a polynomial function, given its equation, by using a variety of strategies (e.g., using the sign of the first derivative; using the sign of the second derivative; identifying even or odd functions) to determine its key features (e.g., increasing/ decreasing intervals, intercepts, local maxima and minima, points of inflection, intervals of concavity), and verify using technology

Sketching a Polynomial Function Given its First and Second Derivative

The first and second derivative of a polynomial function can be used to determine key features of the graph of the function. The second derivative is the derivative of the first derivative. Once the key features of the polynomial function are found, the graph of the function can be sketched.

The graph of a polynomial function can be sketched given the equations or graphs of its first and second derivative.

When sketching the graph of a polynomial function from the graphs of its first and second derivative, translating the graph of a function vertically does not change the graph of the derivative. Conversely, if the graph of the derivative is given, the graph of the original function may be any one of an infinite family of functions that are vertical translations of each other. Knowing one point on the graph of the function will specify which function is required.

The following features of the function, $f(x)$, can be determined and used to sketch the graph:

- The function is odd when $f(-x) = -f(x)$ or even when $f(-x) = f(x)$.
- The function approaches positive or negative infinity for extreme positive or negative x-values.
- For the x-intercepts, set $f(x) = 0$ and solve for x.
- For the y-intercept, set $x = 0$ and solve for $f(x)$.
- Intervals of increase and decrease occur when $f'(x) > 0$ and $f'(x) < 0$, respectively.
- Flattening of the slope, usually local maximums and minimums, occurs when $f'(x) = 0$.
- The points of inflection occur when $f''(x) = 0$.
- The function is concave up when $f''(x) > 0$ and concave down when $f''(x) < 0$.

6. The first derivative of a rational expression is defined by $f'(x) = -\frac{4}{x^2}$.

Which of the following statements about the graph of $f(x)$ is **true**?

A. The function $f(x)$ is concave up for all real values of x.

B. The function $f(x)$ is concave up on the interval $-\infty < x < 0$.

C. The function $f(x)$ is concave down on the interval $0 < x < \infty$.

D. The function $f(x)$ is concave down on the interval $-\infty < x < 0$.

Written Response

7. Using the first and second derivative of the function $f(x) = x^4 - 2x^3$, determine the maximum or minimum point or points and the intervals of concavity.

Written Response

8. Determine the first and second derivatives of the function
$f(x) = x^{10} - 2x^7 + 3x^4 - 6x - 9$.

Use the following information to answer the next question.

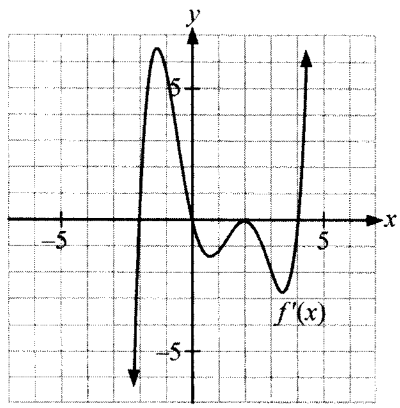
Information about a particular polynomial function, $f(x)$, where $x \in \mathbb{R}$, is provided in the given table.

x	$f(x)$	$f'(x)$	$f''(x)$
-1	$-\frac{5}{3}$	+	−
1	$\frac{5}{3}$	0	−
3	1	0	+
4	$\frac{5}{3}$	$\frac{3}{2}$	+

9. Which of the following conclusions can be made about the graph of $f(x)$ at $x = 4$?
   A. The graph is increasing and concave up.
   B. The graph is decreasing and concave up.
   C. The graph is increasing, and there is a point of inflection.
   D. The graph is decreasing, and there is a point of inflection.

Use the following information to answer the next question.

The graph of $f'(x)$ the derivative of a function $f(x)$, is shown.



10. On approximately which of the following intervals is the graph of $f(x)$ increasing?

A. $-2 < x < 0$ and $4 < x < \infty$

B. $0 < x < 2$ and $5 < x < \infty$

C. $-\infty < x < -1$ and $1 < x < 2$

D. $2.5 < x < 4$ and $5.5 < x < \infty$

Written Response

11. How does knowing the x-intercepts of the graph of the derivative of a polynomial function help to sketch the graph of the function itself?

12. One conclusion that can be reached when the second derivative of a polynomial function is equal to zero at $x = a$ is that the function

A. is concave down at $x = a$

B. has an x-intercept at $x = a$

C. has a local maximum at $x = a$

D. may have a point of inflection at $x = a$

Numerical Response

13. Calculated to the nearest hundredth, the y-coordinate of the local maximum of the function $f(x) = 2x^3 + \frac{1}{2}x^2 - 2x + 9$ is __________.

Use the following information to answer the next question.

A curve can be defined by the following characteristics:

- Odd or even
- y-intercepts and x-intercepts
- Intervals of increase or decrease
- Local extrema
- Points of inflection
- Intervals of concavity

Written Response

14. Explain the curve defined by $f(x) = -\frac{4}{3}x^3 + 4x^2 - 3x$ according to the given characteristics, and use this information to sketch a graph of $f(x)$.

DER2.1 make connections between the concept of motion (i.e., displacement, velocity, acceleration) and the concept of the derivative in a variety of ways (e.g., verbally, numerically, graphically, algebraically)

Making Connections between the Concepts of Motion and Derivatives

If a car is travelling due north on a straight road, the car's speedometer gives its instantaneous velocity in a northward direction at any particular time.

Instantaneous velocity is the instantaneous rate of change of displacement with respect to time. If the driver of the car accelerates, then the instantaneous acceleration is the instantaneous rate of change of velocity with respect to time.

Displacement is defined as the net change in position of an object from start to finish. In other words, displacement refers to how far away an object is when it finishes moving relative to where it started. If displacement from a starting point, s, is defined by a function of time, t so that $s = f(t)$, then the instantaneous velocity, v, at any time is the derivative of the displacement with respect to time. This can be expressed in the formula

$$v = \frac{ds}{dt} = f'(t).$$

The instantaneous acceleration at any time, a, is the instantaneous rate of change of the velocity, which is the derivative of the velocity with respect to time. This is also the second derivative of the displacement function. This can be expressed in the formula $a = \frac{dv}{dt} = \frac{d^2s}{dt^2} = f''(t)$.

15. Which of the following position functions has a constant acceleration?

A. $s(t) = 2t^2 - 9t + 7$

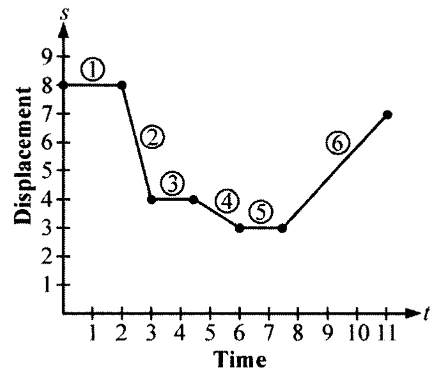
B. $s(t) = 7t^3 - 4t^2 - 3t$

C. $s(t) = 3t^3 - 5t^2 + 4t - 1$

D. $s(t) = 5t^4 - 4t^3 + 4t - 3$

Use the following information to answer the next question.

The displacement-time graph of an object is shown and is divided into six stages.



16. On which stage is the magnitude of the velocity the greatest?

A. 1 **B.** 2

C. 4 **D.** 6

Use the following information to answer the next question.

A rock is thrown vertically upward from ground level with an initial velocity of 30 m/s. At some point, the rock stops travelling upward and falls back toward the ground. The height, h, in metres, of the rock is a function of the time, t, in seconds, defined as $h(t) = -5t^2 + 30t$, where $t \geq 0$.

Numerical Response

17. To the nearest metre, the maximum height reached by the rock is _________ m.

DER2.2 make connections between the graphical or algebraic representations of derivatives and real-world applications (e.g., population and rates of population change, prices and inflation rates, volume and rates of flow, height and growth rates)

CONNECTING REPRESENTATIONS OF DERIVATIVES IN REAL-WORLD SITUATIONS

There are numerous real-world situations where it is important to know and understand historical, current, and predicted rates of change.
Since derivatives are measurements of instantaneous rates of change, working with rates of change often involves analyzing, calculating, estimating, and graphing derivatives.

Example

A particular species of fish grows in length for the first two weeks after birth at a rate measured in millimetres per day given by $L\ (d) = 0.06\pi d^2$, in which d is the number of days after birth.

Explain why $L\ (d)$ is a formula for a derivative and how it is a derivative of a quantity with respect to another quantity.

Solution

The formula is a derivative because it is a rate of change formula.

Because the formula involves a change in length given in millimetres, over a period of time in days, the formula is a derivative of length with respect to time.

What is the degree of the function that has the derivative $L\ (d) = 0.06\pi d^2$?

Solution

The power rule of derivatives states that if $f(x) = x^n$, in which n is a natural number, then the derivative is $f'(x) = nx^{n-1}$. From the power rule of derivatives, the original function has a degree of n, and the derivative function has a degree of $n - 1$, in which n is a natural number. Since the derivative function, $L\ (d) = 0.06\pi d^2$, has a degree of 2, then the degree of the original function is found as follows:

$$2 = n - 1$$
$$n = 3$$

Therefore, the degree of the function that has the derivative $L\ (d) = 0.06\pi d^2$ is 3.

Rounded to the nearest day, what number of days results in a growth rate that is 10 mm/day?

Solution

If the growth rate is 10 mm/day, solve for d when $L\ (d) = 10$.

$$L\ (d) = 0.06\pi d^2$$
$$10 = 0.06\pi d^2$$
$$\frac{10}{0.06\pi} = d^2$$
$$\sqrt{\frac{10}{0.06\pi}} = d$$
$$7 \approx d$$

Therefore, the growth rate is 10 mm/day by the seventh day.

How would doubling the number of days after birth affect the growth rate?

Solution

Determine the growth rate if the number of days were doubled from d to $2d$.

$$L\ (d) = 0.06\pi d^2$$
$$L_2(d) = 0.06\pi(2d)^2$$
$$= 4(0.06\pi d^2)$$
$$= 4L\ (d)$$

Doubling the number of days would quadruple the growth rate.

Use the following information to answer the next question.

Gasoline is leaking from a fuel truck at a rate of $R(t) = 2\,000e^{-0.2t}$ gal/h, where t represents the time in hours.

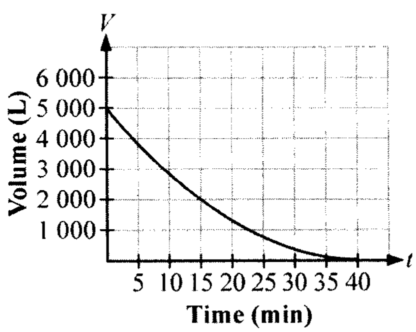
18. The rate of leakage first drops below 225 gal/h after approximately

A. 1.2 h **B.** 2.9 h

C. 5.7 h **D.** 10.9 h

Use the following information to answer the next question.

A tank with a capacity of 5 000 L drains completely in 40 min. The graph provided shows the volume of water remaining in the tank after draining for t min.



19. At which of the following times is the rate of drainage the greatest?

A. 5 min **B.** 15 min

C. 25 min **D.** 35 min

Use the following information to answer the next question.

A bacteria culture grows at a rate proportional to its population. The growth rate, in bacteria/h, of the culture of bacteria is given by the function

$g(t) = 1\,000(2.5)^{\frac{t}{2}}$, in which t is the time measured in hours.

Numerical Response

20. To the nearest whole number, what is the rate of growth of the bacteria culture after 5 h? __________ bacteria/h

*DER2.3 solve problems, using the derivative, that involve instantaneous rates of change, including problems arising from real-world applications (e.g., population growth, radioactive decay, temperature changes, hours of day-light, heights of tides), given the equation of a function**

Solving Instantaneous Rate of Change Problems

If a known function describes a real-world quantity in terms of an independent variable, such as time, the derivative will give the rate of change of that quantity with respect to the independent variable. The derivative can be used to determine the instantaneous rate of change of the quantity for any particular value of the independent variable.

Example

The future size of the population of a particular species of birds in northern Ontario can be estimated by the formula $P = \dfrac{8\,000}{1+53(0.3)^t}$, in which t is the time in years.

Rounded to the nearest whole number, determine the rate of growth of the population after 7 yr using the derivative.

Solution

Find the derivative to determine the rate of growth of the population after t years.

Step 1
In order to find the derivative, write the quotient, $\dfrac{8\,000}{1+53(0.3)^t}$, as a product.

$P = 8\,000\left(1+53(0.3)^t\right)^{-1}$

Step 2
Find the derivative of $P = 8\,000\left(1+53(0.3)^t\right)^{-1}$ using the derivative rules.

The derivative of $(0.3)^t$ is $(0.3)^t \ln 0.3$.

$P = 8\,000\left(1+53(0.3)^t\right)^{-1}$

$\dfrac{dP}{dt} = 8\,000(-1)\left(1+53(0.3)^t\right)^{-2}$

$\left(0+53(0.3)^t \ln 0.3\right)$

$= -8\,000\left(1+53(0.3)^t\right)^{-2}\left(53(0.3)^t \ln 0.3\right)$

Step 3

Calculate $\frac{dP}{dt}$ when $t = 7$.

$\frac{dP}{dt} = -8\,000\left(1 + 53(0.3)^7\right)^{-2}\left(53(0.3)^7 \ln 0.3\right)$

Using a calculator, the result is approximately 109.1.

After 7 yr, the population is growing at a rate of approximately 109 birds/yr.

21. The electric potential (voltage) of a battery that is dying is given by the equation $P = 10 - (0.04)t^2$, where t is the time elapsed in hours. What is the rate of change of the electric potential of the battery after 3 h?

A. −37.5 V/h

B. −0.24 V/h

C. 9.64 V/h

D. 13.23 V/h

Use the following information to answer the next question.

The volume of water in a certain cylindrical tank is increasing at a rate of 80 cm^3/min, and the radius of the tank is 12 cm.

22. The rate of change of the height with respect to time of the given cylindrical tank, to the nearest hundredth, is

A. 0.18 cm/min **B.** 1.06 cm/min

C. 1.75 cm/min **D.** 2.12 cm/min

Use the following information to answer the next question.

Wind chill is the temperature determined from a combination of the actual air temperature and wind speed. For an air temperature of 32, the wind chill, W, in degrees Fahrenheit, is given by the function $W(s) = 55.6 - 22.1s^{0.16}$, where s is the wind speed in mi/h.

Numerical Response

23. To the nearest hundredth, at what rate is the wind chill decreasing when the wind speed is 20 mi/h? __________°F/h

DER2.4 solve optimization problems involving polynomial, simple rational, and exponential functions drawn from a variety of applications, including those arising from real-world situations

SOLVING OPTIMIZATION PROBLEMS

An optimization problem involves finding the preferred or optimum amount of a specific quantity. If a function represents an application involving optimization, the optimum quantity or amount occurs at either the relative maximum or minimum or at the end values of the function.

The following steps can be taken to solve optimization problems:

1. Find the derivative of the function, $f(x)$.
2. Determine the potential local maximum or minimum by setting $f'(x) = 0$.
3. If necessary, such as for maximum revenue questions, substitute the local maximum value into $f(x)$ to determine the maximum value desired.

Use the following information to answer the next question.

A wire with a length of 30 cm is cut into two pieces. One piece is bent into the shape of a square. The other piece is bent into the shape of a rectangle whose length is twice its width. The relationship between the lengths of wire for the two shapes is given by $30 = 4x + 6y$, in which x is the width of the square and y is the width of the rectangle, both in metres. The sum of the areas of the square and the rectangle is given by $A = x^2 + 2y^2$. The sum is also a minimum.

24. What is the length of the longer piece of wire, to the nearest tenth of a metre?

A. 2.6 m **B.** 3.5 m

C. 14.1 m **D.** 15.9 m

Use the following information to answer the next question.

The cost of producing x amount of dolls is defined by the function $C(x) = 4x^2 - 64x + 381$.

Numerical Response

25. How many dolls are produced when the cost is at a minimum? __________

Use the following information to answer the next question.

The equation $P = xy^3$ represents the product of one positive number, x, and the cube of another positive number, y.

Written Response

26. If the sum of the two numbers is 12, find the numbers that would yield a maximum product, P.

Use the following information to answer the next question.

A certain courier company will only accept cylindrical packages whose height and circumference combined do not exceed 10 ft.

Written Response

27. What must the height of the cylindrical package be to maximize its volume if the volume of a cylinder is given by $V = \pi r^2 h$, in which V represents the volume in cubic feet, r represents the radius in feet, and h represents the height in feet?

DER2.5 solve problems arising from real-world applications by applying a mathematical model and the concepts and procedures associated with the derivative to determine mathematical results, and interpret and communicate the results

SOLVING PROBLEMS USING MATHEMATICAL MODELS

It is a common practice in many fields of study to attempt to describe changing situations with mathematical models. Because rates of change or maximums and minimums are often of interest in these situations, the mathematics usually involve finding the derivative. The models are seldom one hundred percent accurate, but they are usually good estimates for particular values of the independent variable.

Example

Factories A and B are 10 km apart. Factory B emits four times the pollutants into the air as factory A. The number, N, of particles of pollutants in the air from a given factory is inversely proportional to the square of the distance from that factory $\left(y = \dfrac{k}{f(x)}\right)$.

John and Irene want to build a house adjacent to a straight road that connects the two factories. They want the house to be located at the point at which the pollution from the factories is the least.

How far from factory A, to the nearest tenth of a kilometre, should John and Irene build their house in order to experience the minimum amount of pollution from the two factories? Explain your answer.

Solution

Step 1

Determine a function that represents the total number of particles emitted by both factories.

- Let x be the distance from the house to factory A.
- Let $(10 - x)$ be the distance from the house to factory B.

The number of particles from factory A is $\dfrac{k}{x^2}$, and the number of particles from factory B is $\dfrac{4k}{(10-x)^2}$.

The total number of particles of pollution at the house would be $N = \dfrac{k}{x^2} + \dfrac{4k}{(10-x)^2}$.

Step 2

Determine the derivative of N with respect to x.

$$N = \frac{k}{x^2} + \frac{4k}{(10-x)^2}$$

$$N = kx^{-2} + 4k(10-x)^{-2}$$

$$\frac{dN}{dx} = -2kx^{-3} + \left(-8k(10-x)^{-3}\right)(-1)$$

$$\frac{dN}{dx} = -2kx^{-3} + 8k(10-x)^{-3}$$

The derivative of N with respect to x is $-2kx^{-3} + 8k(10-x)^{-3}$.

Step 3
Determine the minimum amount of pollution particles using the derivative.
The minimum amount of pollution will occur when $\frac{dN}{dx} = 0$.

$$-2kx^{-3} + 8k(10-x)^{-3} = 0$$
$$8k(10-x)^{-3} = 2kx^{-3}$$
$$\frac{4}{(10-x)^3} = \frac{1}{x^3}$$
$$4x^3 = (10-x)^3$$
$$\sqrt[3]{4}x = 10 - x$$
$$\sqrt[3]{4}x + x = 10$$
$$x(\sqrt[3]{4}+1) = 10$$
$$x = \frac{10}{(\sqrt[3]{4}+1)}$$
$$x \approx 3.9$$

Therefore, John and Irene should build their house approximately 3.9 km from factory A. The answer is a minimum because the pollution would increase as you get closer to either of the factories.
You can see this from the form of the function $N = \frac{k}{x^2} + \frac{4k}{(10-x)^2}$ because as the denominators of the rational expressions approach 0, their values approach infinity.

Use the following information to answer the next question.

A bacterial culture begins with 1 000 bacteria, and after 4 h, the count is 5 000. The bacterial culture grows according to the equation $C(t) = Ae^{kt}$, where A is the initial number of bacteria in the culture, k is a constant, and t is the time in hours.

28. Rounded to the nearest whole number, what is the rate of growth of the bacterial culture after 5 h?

A. 3 008 bacteria/h

B. 3 009 bacteria/h

C. 7 476 bacteria/h

D. 7 477 bacteria/h

Use the following information to answer the next question.

The velocity of a flowing river at a popular picnic area is modelled by the function $v(t) = 32 + 5\sin(\sqrt{t}+5)$, $0 \le t \le 68$, where $v(t)$ is the velocity measured in m/min and t is the time in minutes.

29. If a swimmer jumps into the water and begins swimming against the current at a constant rate of 85 m/min, his rate of acceleration in the water after 10 min will be approximately

A. -28.66 m/min^2

B. -0.48 m/min^2

C. 0.48 m/min^2

D. 56.34 m/min^2

Use the following information to answer the next question.

After having been shaped by using heat, a piece of metal with a temperature of 1 500°C must be dropped into a tank of water to cool. The tank of water is kept at a temperature of 35°C. After 10 min in the cooling tank, the metal's temperature drops to 750°C. The rate of decrease in the metal's temperature is said to be negligible when the metal reaches 75°C, so the metal is removed from the tank when it reaches this temperature.
The change in temperature of the piece of metal follows Newton's law of cooling, which states that the difference in temperature between an object and its surroundings decreases exponentially according to the relationship $T - T_s = (T_0 - T_s)e^{kt}$, where T is the object's temperature at time t, T_s is the temperature of the surroundings, T_0 is the initial temperature of the object, and k is a constant representing the relative rate of cooling of the given object.

Numerical Response

30. To the nearest hundredth, what is the negligible rate of temperature decrease for this particular metal? _________°C/min

Answers and Solutions

Derivatives and their Applications

1. C	7. WR	13. 9.96	19. A	25. 8
2. C	8. WR	14. WR	20. 9882	26. WR
3. WR	9. A	15. A	21. B	27. WR
4. WR	10. A	16. B	22. A	28. A
5. C	11. WR	17. 45	23. 0.29	29. C
6. D	12. D	18. D	24. D	30. 2.87

1. C

On the graph of $f(x)$, there is an abrupt change in slope at $x = -1$. Therefore, the derivative graph will be undefined at $x = -1$.

To the left of $x = -1$, the slope is constant at $\frac{4-0}{-4.3-(-1)} \approx -1.2$. Therefore, on the derivative graph $f'(x)$, there is a horizontal line at $y = -1.2$ defined on the interval $-\infty < x < -1$.

To the right of $x = -1$, the slope is constant at $\frac{4-0}{2.3-(-1)} \approx 1.2$. Therefore, on the derivative graph $f'(x)$, there is a horizontal line at $y = 1.2$ defined on the interval $-1 < x < \infty$.

Graph C best represents the graph of $f'(x)$.

2. C

Analyze the y-values from left to right of the graph of the derivative.

- $-\infty < x < 0$, y-values are negative and increasing
- $x = 0$, y-value is zero
- $0 < x < 1.4$, y-values are negative and decreasing
- $1.4 < x < 2$, y-values are negative and increasing
- $x = 2$, y-value is zero
- $2 < x < \infty$, y-values are positive and increasing

Since the y-values of the graph of the derivative are equal to the slopes of $f(x)$, the following slope values should occur in the graph of $f(x)$:

- $-\infty < x < 0$, slope is negative and increasing
- $x = 0$, slope is zero
- $0 < x < 1.4$, slope is negative and decreasing
- $1.4 < x < 2$, slope is negative and increasing
- $x = 2$, slope is zero
- $2 < x < \infty$, slope is positive and increasing

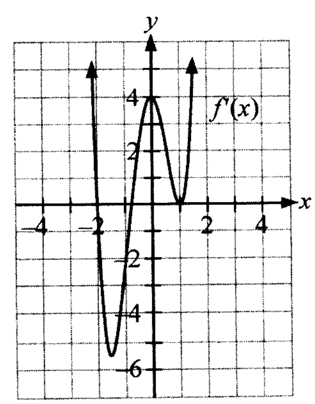
The graph in alternative C is the only graph that shows these slope values.

3. WR

Step 1
Analyze the slope of the graph, when it is viewed from left to right, using approximate x-values.

- $x < -2$, the slope is positive and decreasing
- $x = -2$, the slope is 0
- $-2 < x < -1.4$, the slope is negative and decreasing
- $x = -1.4$ is a point of inflection
- $-1.4 < x < -0.9$, the slope is negative and increasing
- $x = -0.9$, the slope is 0
- $-0.9 < x < 0$, the slope is positive and increasing
- $x = 0$ is a point of inflection
- $0 < x < 1$, the slope is positive and decreasing
- $x = 1$, the slope is 0
- $x > 1$, the slope is positive and increasing

Step 2
Graph the approximate slope values. Connect the points using a thick line. The resulting graph is the derivative of the function.



4. WR

The function $f(x)$ may be undefined at $x = a$. The graph would have a vertical asymptote or a hole at $x = a$.

The derivative of the function may be undefined at $x = a$. The graph might have a cusp or an endpoint at $x = a$.

The function may be a piecewise function. The graph of $f(x)$ would show a break and not be continuous at $x = a$.

5. C

The instantaneous rate of change is equal to 0 when the slope of the tangent is equal to 0 on the graph of $f(x)$. At approximately $x = 2.0$, the slope of the tangent is equal to 0.

Therefore, the instantaneous rate of change of $f(x)$ is equal to 0 at approximately $x = 2.0$.

6. D

Step 1
Determine the second derivative of the function $f(x)$.

$$f'(x) = -\frac{4}{x^2}$$
$$f'(x) = -4x^{-2}$$
$$f''(x) = 8x^{-3}$$

Step 2
Determine the x-coordinate of the inflection point.

Solve for x when $f''(x) = 0$.

$$f''(x) = 8x^{-3}$$
$$0 = 8x^{-3}$$
$$0 = x^{-3}$$
$$x = 0$$

Step 3
Determine the intervals of concavity.

Substitute any value greater than 0 into $f''(x)$, and solve.

Let $x = 1$.

$$f''(x) = 8x^{-3}$$
$$f''(1) = 8(1)^{-3} = 8$$

The function $f''(x)$ is positive on the interval $x > 0$. Therefore, $f(x)$ is concave up on the interval $0 < x < \infty$.

Substitute any value less than 0 into $f''(x)$, and solve.

Let $x = -1$.

$$f''(x) = 8x^{-3}$$
$$f''(-1) = 8(-1)^{-3} = -8$$

The function $f''(x)$ is negative on the interval $x < 0$. Therefore, $f(x)$ is concave down on the interval $-\infty < x < 0$.

7. WR

Determine the values of x when $f'(0) = 0$.

$$f(x) = x^4 - 2x^3$$
$$f'(x) = 4x^3 - 6x^2$$
$$0 = 2x^2(2x - 3)$$
$$x = 0, \frac{3}{2}$$

Using test points on either side of $x = 0$ and $x = \frac{3}{2}$ reveals that the first derivative is negative on the interval $-\infty < x < \frac{3}{2}$ and positive on the interval $\frac{3}{2} < x < \infty$. A negative first derivative on a given interval corresponds to a decreasing function on that interval. Similarly, a positive first derivative on a given interval corresponds to an increasing function on that interval. Therefore, the graph of $f(x)$ is decreasing on the interval $-\infty < x < \frac{3}{2}$ and increasing on the interval $\frac{3}{2} < x < \infty$.

In addition, at the point where the first derivative of a function is equal to zero and changes sign from negative to positive, the function has a local minimum. For the function $f(x)$, a local minimum occurs at $x = \frac{3}{2}$ of $f\left(\frac{3}{2}\right) = -\frac{27}{16}$.

Determine the values of x when $f''(x) = 0$.

$$f'(x) = 4x^3 - 6x^2$$
$$f''(x) = 12x^2 - 12x$$
$$0 = 12x(x - 1)$$
$$x = 0, 1$$

Using test points on either side of $x = 0$ and $x = 1$ reveals that the second derivative is negative on the interval $0 < x < 1$ and positive on the intervals $-\infty < x < 0$ and $1 < x < \infty$. A negative second derivative on a given interval corresponds to a function that is concave down on that interval. Similarly, a positive second derivative on a given interval corresponds to function that is concave up on that interval. Therefore, the graph of $f(x)$ is concave down on $0 < x < 1$ and concave up on $-\infty < x < 0$ and $1 < x < \infty$.

In addition, at the point where the second derivative of a function is equal to zero and changes sign from negative to positive or positive to negative, the graph of the function moves from concave down to concave up or concave up to concave down, resulting in points of inflection where $f''(x) = 0$. On the graph of $f(x)$, points of inflection occur at $x = 0$ and $x = 1$ of $(0, f(0)) = (0, 1)$ and $(1, f(1)) = (1, -1)$.

8. WR

Step 1
Determine the first derivative of the function.

$f(x) = x^{10} - 2x^7 + 3x^4 - 6x - 9$
$f'(x) = 10x^9 - 14x^6 + 12x^3 - 6$

Step 2
Determine the second derivative of the function.

$f'(x) = 10x^9 - 14x^6 + 12x^3 - 6$
$f''(x) = 90x^8 - 84x^5 + 36x^2$

9. A

At $x = 4$, $f'(x)$ is positive; therefore, the function $f(x)$ is increasing.

At $x = 4$, $f''(x)$ is also positive; therefore, the function $f(x)$ is concave up at $x = 4$.

10. A

The graph of $f(x)$ is increasing on intervals where the graph of $f'(x)$ is positive.

Observing the given graph, $f'(x)$ is positive on the intervals $-2 < x < 0$ and $4 < x < \infty$.

Therefore, the graph of $f(x)$ is increasing on the intervals $-2 < x < 0$ and $4 < x < \infty$.

11. WR

The coordinates of the x-intercepts of the derivative function correspond to the x-coordinates of local extrema on the graph of the function itself.
More information is required about the first derivative in order to conclude whether the extrema are local maxima or local minima.

12. D

When the second derivative function is equal to zero at a particular point, then there is a point of inflection on the original function at that point. Thus, when the second derivative graph of a polynomial function is equal to zero at $x = a$, there is a point of inflection on the function graph at $x = a$.

13. 9.96

Step 1
Find the first derivative of the function $f(x)$.

$f(x) = 2x^3 + \frac{1}{2}x^2 - 2x + 9$
$f'(x) = 6x^2 + x - 2$

Step 2
Determine the intervals where the function is increasing and decreasing.
Solve for x when $f(x) = 0$.

$$\begin{aligned} f'(x) &= 6x^2 + x - 2 \\ 0 &= 6x^2 + x - 2 \\ 0 &= 6x^2 + 4x - 3x - 2 \\ 0 &= 2x(3x + 2) - 1(3x + 2) \\ 0 &= (2x - 1)(3x + 2) \end{aligned}$$

$2x - 1 = 0$
$x = \frac{1}{2}$

$3x + 2 = 0$
$x = \frac{-2}{3}$

Test interval $-\infty < x < \frac{-2}{3}$.

$f'(x) = 6x^2 + x - 2$
$f'(-1) = 6(-1)^2 + (-1) - 2$
$= 3$

Test interval $\frac{-2}{3} < x < \frac{1}{2}$.

$f'(x) = 6x^2 + x - 2$
$f'(0) = 6(0)^2 + (0) - 2$
$= -2$

Test interval $\frac{1}{2} < x < \infty$.

$f'(x) = 6x^2 + x - 2$
$f'(1) = 6(1)^2 + (1) - 2$
$= 5$

The given table summarizes the behaviour of the function.

Interval	$f'(x)$	Increasing / Decreasing
$-\infty < x < \frac{-2}{3}$	+	Increasing
$\frac{-2}{3} < x < \frac{1}{2}$	–	Decreasing
$\frac{1}{2} < x < \infty$	+	Increasing

Step 3
Determine the local maximum point.

The derivative is positive to the left of $x = -\frac{2}{3}$ and negative to the right of $x = -\frac{2}{3}$. Therefore, there is a local maximum at $x = -\frac{2}{3}$.

Solve $f(x)$ when $x = -\frac{2}{3}$.

$$f(x) = 2x^3 + \frac{1}{2}x^2 - 2x + 9$$
$$f\left(-\frac{2}{3}\right) = 2\left(-\frac{2}{3}\right)^3 + \frac{1}{2}\left(-\frac{2}{3}\right)^2 - 2\left(-\frac{2}{3}\right) + 9$$
$$\approx 9.96$$

Therefore, the y-coordinate of the local maximum of the function $f(x)$ is 9.96.

14. WR

Since the curve is a cubic function with a negative leading coefficient, the graph will begin in the second quadrant (for infinitely small values of x, $f(x)$ is very large), and the graph will end in the fourth quadrant (for infinitely large values of x, $f(x)$ is very small).

Step 1
Test the function to determine if it is odd or even.
Rewrite the equation when $f(-x)$.

$$f(x) = -\frac{4}{3}x^3 + 4x^2 - 3x$$
$$f(-x) = -\frac{4}{3}(-x)^3 + 4(-x)^2 - 3(-x)$$
$$f(-x) = \frac{4}{3}x^3 + 4x^2 + 3x$$

Since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, the function is neither odd nor even.

Step 2
Determine the coordinates of the y intercept.

$$f(x) = -\frac{4}{3}x^3 + 4x^2 - 3x$$
$$f(0) = -\frac{4}{3}(0)^3 + 4(0)^2 - 3(0)$$
$$f(0) = 0$$

The y-intercept is at (0, 0).
Determine the coordinates of any x-intercepts.

$$f(x) = -\frac{4}{3}x^3 + 4x^2 - 3x$$
$$0 = -\frac{4}{3}x^3 + 4x^2 - 3x$$
$$0 = -4x^3 + 12x^2 - 9x$$
$$0 = -x(4x^2 - 12x + 9)$$
$$0 = -x(2x - 3)(2x - 3)$$

$f(x) = 0$ when $x = 0, \frac{3}{2}$. Therefore, the x-intercepts are at (0, 0), $\left(\frac{3}{2}, 0\right)$.

Step 3
Determine intervals of increase/decrease and local extrema.
Solve for x when $f'(x) = 0$.

$$f(x) = -\frac{4}{3}x^3 + 4x^2 - 3x$$
$$f'(x) = -4x^2 + 8x - 3$$
$$0 = (-2x + 1)(2x - 3)$$
$$x = \frac{1}{2}, \frac{3}{2}$$

Interval	$f'(x)$	$f(x)$
$-\infty < x < \frac{1}{2}$	–	Decreasing
$\frac{1}{2} < x < \frac{3}{2}$	+	Increasing
$\frac{3}{2} < x < \infty$	–	Decreasing

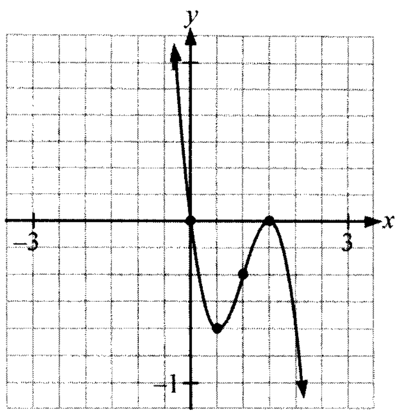
There is a local minimum at $\left(\frac{1}{2}, f\left(\frac{1}{2}\right)\right) = \left(\frac{1}{2}, -\frac{2}{3}\right)$ and a local maximum at $\left(\frac{3}{2}, f\left(\frac{3}{2}\right)\right) = \left(\frac{3}{2}, 0\right)$.

Step 4
Determine the point of inflection and the intervals of concavity.
Solve for x when $f''(x) = 0$.

$$f'(x) = -4x^2 + 8x - 3$$
$$f''(x) = -8x + 8$$
$$0 = -8(x - 1)$$
$$x = 1$$

The function has a point of inflection at $(1, f(1)) = \left(1, -\frac{1}{3}\right)$. Using test points in the regions $-\infty < x < 1$ and $1 < x < \infty$, the function is concave up on $-\infty < x < 1$ and concave down on $1 < x < \infty$.

Step 5
Use the characteristics identified to graph an approximate solution.



15. A

When given the position function of an object, the acceleration of the object can be found by taking the second derivative of the position function.
To determine which of the given position functions would yield a constant acceleration, find the second derivative (acceleration function) of each position function.

Determine the second derivative of $s(t) = 2t^2 - 9t + 7$.

$$s(t) = 2t^2 - 9t + 7$$
$$s'(t) = v(t) = 4t - 9$$
$$s''(t) = v'(t) = a(t) = 4$$

Determine the second derivative of $s(t) = 7t^3 - 4t^2 - 3t$.

$$s(t) = 7t^3 - 4t^2 - 3t$$
$$s'(t) = v(t) = 21t^2 - 8t - 3$$
$$s''(t) = v'(t) = a(t) = 42t - 8$$

Determine the second derivative of $s(t) = 3t^3 - 5t^2 + 4t - 1$.

$$s(t) = 3t^3 - 5t^2 + 4t - 1$$
$$s'(t) = v(t) = 9t^2 - 10t + 4$$
$$s''(t) = v'(t) = a(t) = 18t - 10$$

Determine the second derivative of $s(t) = 5t^4 - 4t^3 + 4t - 3$.

$$s(t) = 5t^4 - 4t^3 + 4t - 3$$
$$s'(t) = v(t) = 20t^3 - 12t^2 + 4$$
$$s''(t) = v'(t) = a(t) = 60t^2 - 24t$$

The function $s(t) = 2t^2 - 9t + 7$ has a second derivative with a constant value; therefore, it has a constant acceleration.

16. B

The velocity of an object is the instantaneous rate of change of the object's displacement with respect to time. In other words, the first derivative graph of a displacement-time graph is a velocity-time graph.

In this case, the velocity-time graph is made up of six polynomial elements to correspond with the six polynomial elements of the displacement-time graph. The degree of each element in the velocity-time graph will be 1 degree less than the corresponding element of the displacement-time graph.

The velocity during each stage of the graph is described as follows:

Stage 1
The displacement is constant. In other words, the rate of change of the displacement at this stage is zero. Therefore, the velocity over the same time interval will be zero.

Stage 2
The displacement is decreasing in a linear fashion. This line is a polynomial of degree 1. Therefore, the velocity over the same interval will be constant (degree 0) and negative.
The rate of change of the displacement at this stage is approximately −4. Therefore, the velocity at this stage is approximately −4.

Stage 3
The displacement is constant. In other words, the rate of change of the displacement at this stage is zero. Therefore, the velocity over the same time interval will be zero.

Stage 4
The displacement is decreasing in a linear fashion. This line is a polynomial of degree 1. Therefore, the velocity over the same interval will be constant (degree 0) and negative.

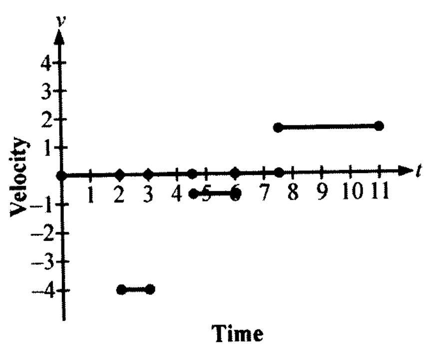
The rate of change of the displacement at this stage is approximately $-\frac{1}{2}$. Therefore, the velocity at this stage is approximately $-\frac{1}{2}$.

Stage 5
The displacement is constant. In other words, the rate of change of the displacement at this stage is zero. Therefore, the velocity over the same time interval will be zero.

Stage 6
The displacement is increasing in a linear fashion. This line is a polynomial of degree 1. Therefore, the velocity over the same interval will be constant (degree 0) and positive.

The rate of change of the displacement at this stage is approximately $\frac{4}{3}$. Therefore, the velocity at this stage is approximately $\frac{4}{3}$.

The velocity-time graph that relates to the given displacement-time graph is shown.



From the velocity-time graph, the magnitude of the velocity is greatest on the interval $2 < t < 3$ which corresponds to stage 2.

17. 45

Because the height function, $h(t) = -5t^2 + 30t$, has a negative a-value in a quadratic function of the form $h(t) = at^2 + bt + c$, it is a parabola that opens downward. Therefore, the function reaches a maximum (rather than a minimum).

The maximum height occurs when the rock stops travelling, just before it begins to fall toward the ground. The velocity of the rock at its maximum height is 0 m/s.

Step 1
Determine when the rock reaches its maximum height.
Find the velocity function (the first derivative of the position function), set it equal to 0, and solve for t.

$$\begin{aligned} h(t) &= -5t^2 + 30t \\ h'(t) = v(t) &= -10t + 30 \\ 0 &= -10t + 30 \\ t &= 3 \end{aligned}$$

Therefore, the rock reaches its maximum height after 3 s.

Step 2
Determine the maximum height of the rock.
To determine the maximum height reached by the rock, evaluate the height function when $t = 3$, i.e., $h(3)$.

$$\begin{aligned} h(t) &= -5t^2 + 30t \\ h(3) &= -5(3)^2 + 30(3) \\ &= -45 + 90 \\ &= 45 \end{aligned}$$

The maximum height reached by the rock is 45 m after 3 s.

18. D

The rate of change of the amount of gasoline in a fuel truck is given by $R(t) = 2\,000e^{-0.2t}$ gal/h.

Solve for t when $R(t) = 225$.

$$\begin{aligned} 225 &= 2\,000e^{-0.2t} \\ \frac{225}{2\,000} &= e^{-0.2t} \\ \ln\left(\frac{225}{2\,000}\right) &= \ln e^{-0.2t} \\ \ln\left(\frac{225}{2\,000}\right) &= -0.2t\ln e \\ \ln\left(\frac{225}{2\,000}\right) &= -0.2t \\ -\frac{1}{0.2}\ln\left(\frac{225}{2\,000}\right) &= t \\ 10.9 &= t \end{aligned}$$

The rate of leakage first drops below 225 gal/h after approximately 10.9 h.

19. A

On the volume-time graph, the instantaneous rate of drainage is shown by the slope of the line tangent to the curve at any given instant within the draining period. As the curve indicates, the rate of drainage is greatest immediately after drainage begins (at the steepest part of the curve). With time, the rate slows as the volume in the tank decreases (as the curve flattens out).

Therefore, compared with the other times given, the rate of drainage at 5 min is the greatest.

20. 9882

To determine the rate of growth of the bacteria after 5 h, evaluate $g(5)$.

$$g(t) = 1\,000(2.5)^{\frac{t}{2}}$$

$$\begin{aligned} g(5) &= 1\,000(2.5)^{\frac{5}{2}} \\ &= 1\,000(2.5)^{2.5} \\ &= 9\,882 \end{aligned}$$

The growth rate is 9 882 bacteria/h.

21. B

If the electric potential of a dying battery is given by the function $P = 10 - (0.04)t^2$, then the rate of change of the battery's electric potential after 3 h can be found by evaluating the derivative of the given function at $t = 3$.

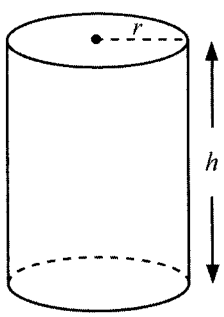
$$\begin{aligned} P &= 10 - (0.04)t^2 \\ \frac{dP}{dt} &= -(0.04)(2t) \\ &= -0.08t \\ \left.\frac{dp}{dt}\right|_{t=3} &= -0.08(3) \\ &= -0.24 \end{aligned}$$

The rate of change of the electric potential of the dying battery after 3 h is -0.24 V/h.

22. A

Step 1

The volume of a cylinder is given by the formula $V = \pi r^2 h$ in which r is the radius, and h is the height.



For this particular situation, it is given that the radius is 12 cm. Since the radius of a right circular cylinder is constant, the given radius may be immediately inserted into the volume function.

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi(12)^2 h \\ &= 144\pi h \end{aligned}$$

Step 2

Determine the rate of change function of the height of water in the tank with respect to time.

Differentiate the volume function with respect to time.

$$\begin{aligned} V &= 144\pi h \\ \frac{dV}{dt} &= 144\pi\frac{dh}{dt} \end{aligned}$$

Step 3

Determine the rate of change of the height of water in the tank with respect to time, given that the rate of volume increase is 80 cm^3/min.

Replace $\frac{dV}{dt}$ with 80 cm^3/min, and solve for $\frac{dh}{dt}$.

$$\begin{aligned} \frac{dV}{dt} &= 144\pi\frac{dh}{dt} \\ 80 &= 144\pi\frac{dh}{dt} \\ \frac{80}{144\pi} &= \frac{dh}{dt} \\ 0.18 &\approx \frac{dh}{dt} \end{aligned}$$

The rate of change of water height with respect to time is approximately 0.18 cm/min.

23. 0.29

Step 1

Determine the derivative function $W'(s)$.

$$\begin{aligned} W(s) &= 55.6 - 22.1s^{0.16} \\ W'(s) &= -22.1(0.16)s^{-0.84} \\ &= -3.536s^{-0.84} \end{aligned}$$

Step 2

Determine the rate of change of the wind chill when the wind speed is 20 mi/h.

Evaluate for $W'(s)$ when $s = 20$.

$$\begin{aligned} W'(s) &= -3.536s^{-0.84} \\ W'(20) &= -3.536(20)^{-0.84} \\ &\approx -0.29 \end{aligned}$$

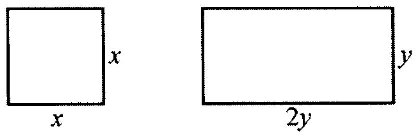
Therefore, the wind chill is decreasing at a rate of approximately 0.29°F/h when the wind speed is 20 mi/h.

24. D

Step 1

Write down the information that is provided, and draw a diagram to illustrate the problem.

The relationship between the lengths of wire is given by $30 = 4x + 6y$. The sum of the areas of the square and rectangle, $A = x^2 + 2y^2$, is to be a minimum.



Step 2

Solve for y in terms of x using the relationship $30 = 4x + 6y$.

$$\begin{aligned} 30 &= 4x + 6y \\ 30 - 4x &= 6y \\ \frac{30 - 4x}{6} &= y \\ 5 - \frac{2}{3}x &= y \end{aligned}$$

Substitute the resulting expression for y into the area formula.

$$\begin{aligned} A &= x^2 + 2y^2 \\ &= x^2 + 2\left(5 - \frac{2}{3}x\right)^2 \\ &= x^2 + 2\left(5 - \frac{2}{3}x\right)\left(5 - \frac{2}{3}x\right) \\ &= x^2 + 2\left(25 - \frac{10}{3}x - \frac{10}{3}x + \frac{4}{9}x^2\right) \\ &= x^2 + 2\left(25 - \frac{20}{3}x + \frac{4}{9}x^2\right) \\ &= x^2 + 50 - \frac{40}{3}x + \frac{8}{9}x^2 \\ &= \frac{17}{9}x^2 - \frac{40}{3}x + 50 \end{aligned}$$

Step 3

A minimum of a function occurs when its derivative is equal to zero and the function is decreasing on the left of the zero and increasing on the right.

$$\begin{aligned} A &= \frac{17}{9}x^2 - \frac{40}{3}x + 50 \\ A' &= \frac{34}{9}x - \frac{40}{3} \\ 0 &= \frac{34}{9}x - \frac{40}{3} \\ \frac{40}{3} &= \frac{34}{9}x \\ \frac{360}{102} &= x \\ \frac{60}{17} &= x \end{aligned}$$

Confirm the minimum as follows.

Interval	$A'(x)$	$A(x)$
$0 < x < \frac{60}{17}$	–	Decreasing
$\frac{60}{17} < x < \frac{30}{4}$	+	Increasing

When $x = \frac{60}{17}$, the sum of the areas is a minimum.

Step 4

Determine the value of y when $x = \frac{60}{17}$.

$$\begin{aligned} 5 - \frac{2}{3}x &= y \\ 5 - \frac{2}{3}\left(\frac{60}{17}\right) &= y \\ \frac{85}{17} - \frac{40}{17} &= y \\ \frac{45}{17} &= y \end{aligned}$$

Step 5

Determine the length of the two wires.

Given the relationship $30 = 4x + 6y$, the two lengths of wire will be as follows.

$$\begin{aligned} 4x &= 4\left(\frac{60}{17}\right) \\ &= \frac{240}{17} \\ &\approx 14.1 \text{ m} \end{aligned}$$

$$\begin{aligned} 6y &= 6\left(\frac{45}{17}\right) \\ &= \frac{270}{17} \\ &\approx 15.9 \text{ m} \end{aligned}$$

Therefore, the longer piece of wire will be approximately 15.9 m.

25. 8

The cost function $C(x) = 4x^2 - 64x + 381$ is a quadratic function that opens upward and has a minimum value.

To determine the amount of dolls produced at minimum cost, find the first derivative, $C'(x)$, set it equal to 0, and solve for x.

$$C(x) = 4x^2 - 64x + 381$$
$$C'(x) = 8x - 64$$
$$0 = 8x - 64$$
$$x = 8$$

The cost is at a minimum when the number of dolls produced is 8.

26. WR

Step 1
Write an equation for the relationship between x and y.
The sum of two numbers, x and y, is 12. An equation that represents this relationship is $12 = x + y$.

Step 2
Isolate x in the sum equation.

$$12 = x + y$$
$$12 - y = x$$

Substitute the expression for x into the product formula, $P = xy^3$.

$$P = xy^3$$
$$= (12 - y)y^3$$
$$= 12y^3 - y^4$$

Step 3
A maximum of a function occurs where its derivative is equal to zero and the function is increasing on the left of the zero and decreasing on the right. Find the first derivative of the function, and solve for y when $P' = 0$.

$$P = 12y^3 - y^4$$
$$P' = 36y^2 - 4y^3$$
$$0 = 4y^2(9 - y)$$
$$y = 0 \text{ or}$$
$$y = 9$$

Step 4
Determine the numbers that yield the maximum product.
Since the goal is a maximum product, y cannot be zero. Thus, only consider $y = 9$.
Using the sum equation, determine the value of x when $y = 9$.

$$x = 12 - y$$
$$= 12 - 9$$
$$= 3$$

Therefore, the numbers that yield a maximum product are 3 and 9.

27. WR

To determine the height, h, of a cylinder that results in a maximum volume, the radius, r, must be written in terms of h (or vice versa) in the volume formula $V = \pi r^2 h$.

Step 1
Relate the variables h and r.
While the formula for the volume of a cylinder is given, the relationship between r and h must be established using the information that $h + C = 10$ ft, in which C is the circumference of the cylinder's base and h is the height.
Since $C = 2\pi r$,

$$h + C = 10$$
$$h + 2\pi r = 10$$
$$h = 10 - 2\pi r$$

In this case, it is easier to find h in terms of r.

Step 2
Substitute $10 - 2\pi r$ for h into the volume formula.

$$V = \pi r^2 h$$
$$V = \pi r^2(10 - 2\pi r)$$
$$V = 10\pi r^2 - 2\pi^2 r^3$$

Step 3
Determine the derivative of the volume function. Set it equal to zero, and solve for r.

$$V = 10\pi r^2 - 2\pi^2 r^3$$
$$\frac{dV}{dr} = 20\pi r - 6\pi^2 r^2$$
$$0 = 20\pi r - 6\pi^2 r^2$$
$$0 = 2\pi r(10 - 3\pi r)$$
$$r = 0, \frac{10}{3\pi}$$

Step 4
Test the validity of r.
A radius of 0 ft is not a possibility because that would result in a volume of 0 ft^3. Test only $r = \frac{10}{3\pi}$ to see if this radius yields a maximum volume.

Interval	$\frac{dV}{dr}$	V
$0 < r < \frac{10}{3\pi}$	+	Increasing
$\frac{10}{3\pi} < r < \frac{10}{2\pi}$	–	Decreasing

Therefore, the volume is a maximum when $r = \frac{10}{3\pi}$.

Step 5
Determine the height of the cylinder.
The height of a cylinder with a maximum volume is as follows:

$$\begin{aligned} h &= 10 - 2\pi r \\ &= 10 - 2\pi\left(\frac{10}{3\pi}\right) \\ &= 10 - \frac{20}{3} \\ &= \frac{10}{3}\text{ ft} \end{aligned}$$

28. A

Step 1
Using the given information, determine the exponential function that represents the number of bacteria in the culture.
The number of bacteria in the culture is given by the equation $C(t) = Ae^{kt}$.
At $t = 0$, the culture contains 1 000 bacteria.
Solve for A.

$$\begin{aligned} C(t) &= Ae^{kt} \\ C(0) &= Ae^{k(0)} \\ 1\,000 &= A \end{aligned}$$

After 4 h, the culture contains 5 000 bacteria; therefore, $C(4) = 5\,000$.
Solve for k.

$$\begin{aligned} C(t) &= Ae^{kt} \\ C(4) &= 1\,000e^{4k} \\ 5\,000 &= 1\,000e^{4k} \\ 5 &= e^{4k} \\ \ln 5 &= 4k \\ \frac{1}{4}\ln 5 &= k \end{aligned}$$

Therefore, the exponential function that represents the number of bacteria in the culture is $C(t) = 1\,000e^{\ln 5\left(\frac{t}{4}\right)}$.

Step 2
Determine the rate of growth of the bacterial culture after 5 h.
Find the derivative function $C'(t)$, and evaluate when $t = 5$.

$$\begin{aligned} C(t) &= 1\,000e^{\ln 5\left(\frac{t}{4}\right)} \\ C'(t) &= 1\,000e^{\ln 5\left(\frac{t}{4}\right)} \cdot \left(\frac{1}{4}\right)\ln 5 \\ C'(5) &= 1\,000e^{\ln 5\left(\frac{5}{4}\right)} \cdot \left(\frac{1}{4}\right)\ln 5 \\ &\approx 3\,008 \end{aligned}$$

Therefore, the rate of growth of the bacterial culture after 5 h is 3 008 bacteria/h.

29. C

Step 1

Determine the swimmer's actual velocity.

If the swimmer swims against the current at a rate of 85 m/min, his actual velocity in the water, $V(t)$, will be the difference between his swimming rate and the velocity of the river's water flow.

$$\begin{aligned} V(t) &= 85 - v(t) \\ &= 85 - (32 + 5\sin(\sqrt{t+5})) \\ &= 85 - 32 - 5\sin\sqrt{t+5} \\ &= 53 - 5\sin\sqrt{t+5} \end{aligned}$$

Step 2

Determine the swimmer's acceleration in the water after 10 min of swimming.

Find the derivative of the function $V(t) = 53 - 5\sin\sqrt{t+5}$, and evaluate when $t = 10$.

Make sure your calculator is in radian mode.

$$V(t) = 53 - 5\sin\sqrt{t+5}$$

$$V(t) = 53 - 5\sin(t+5)^{\frac{1}{2}}$$

$$V'(t) = a(t) = -5\cos(t+5)^{\frac{1}{2}}\left(\frac{1}{2}\right)(t+5)^{-\frac{1}{2}}$$

$$= -\frac{5}{2\sqrt{t+5}}\cos\sqrt{t+5}$$

$$a(10) = -\frac{5}{2\sqrt{10+5}}\cos\sqrt{10+5}$$

$$\approx 0.48$$

After 10 min of swimming, the swimmer is accelerating in the water at a rate of approximately 0.48 m/min^2.

30. 2.87

Step 1

Determine the exponential function that represents the object's temperature after time t.

Substitute known values into the formula for Newton's law of cooling.

$$T - T_s = (T_0 - T_s)e^{kt}$$

$$750 - 35 = (1\,500 - 35)e^{10k}$$

$$715 = 1\,465e^{10k}$$

Solve for the constant, k.

$$715 = 1\,465e^{10k}$$

$$\frac{715}{1\,465} = e^{10k}$$

$$\frac{143}{293} = e^{10k}$$

$$\ln\left(\frac{143}{293}\right) = 10k$$

$$\frac{1}{10}\ln\left(\frac{143}{293}\right) = k$$

Therefore, at time t, the object's temperature can be determined by using the following function:

$$T - 35 = (1\,500 - 35)e^{kt}$$

$$T = 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)} + 35$$

Step 2

Determine the time, t, when $T = 75$°C.

$$T = 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)} + 35$$

$$75 = 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)} + 35$$

$$40 = 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)}$$

$$\frac{40}{1\,465} = e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)}$$

$$\ln\left(\frac{40}{1\,465}\right) = \ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)$$

$$\frac{\ln\left(\frac{40}{1\,465}\right)}{\ln\left(\frac{143}{293}\right)} = \frac{t}{10}$$

$$\frac{10\ln\left(\frac{40}{1\,465}\right)}{\ln\left(\frac{143}{293}\right)} = t$$

$$50.196 \text{ min} \approx t$$

Step 3

Find the rate of change of the metal's temperature after 50.196 min.

Take the derivative of the function $T = 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)} + 35$, and evaluate T' when $t = 50.196$.

$$T = 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)} + 35$$

$$T' = 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{t}{10}\right)} \cdot \left(\frac{1}{10}\right)\ln\left(\frac{143}{293}\right)$$

$$T'(50.196)$$
$$= 1\,465e^{\ln\left(\frac{143}{293}\right)\left(\frac{50.196}{10}\right)} \cdot \left(\frac{1}{10}\right)\ln\left(\frac{143}{293}\right)$$
$$\approx -2.87$$

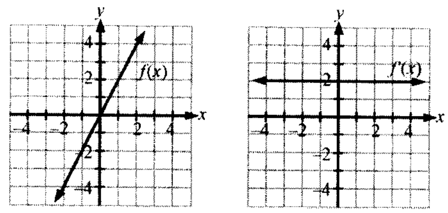
The rate of temperature decrease of the metal is approximately 2.87°C/min when the temperature of the metal reaches 75°C.

Unit Test — Derivatives and their Applications

Use the following information to answer the next question.

Four pairs of graphs are shown.

1.



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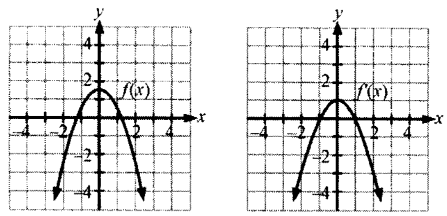

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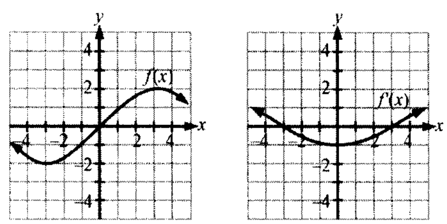

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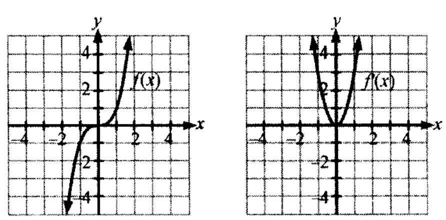


1. Which pairs of graphs represent the graph of a function and the graph of its derivative?

A. 1 and 4 **B.** 1 and 2

C. 3 and 4 **D.** 2 and 3

Use the following information to answer the next multipart question.

2. A function, $f(x)$, is cubic and has a positive leading coefficient.

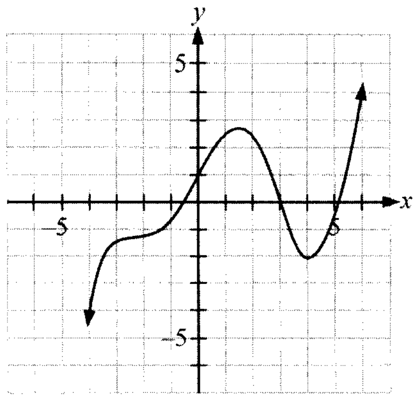
Written Response

a) Describe the shape and orientation of the graph of $f'(x)$.

b) If the graph of $f(x)$ was transformed by a reflection in the x-axis to become the graph of $g(x)$, describe the differences, if any, that would be evident between the graphs of $f'(x)$ and $g'(x)$.

Use the following information to answer the next question.

The graph of a function is shown.

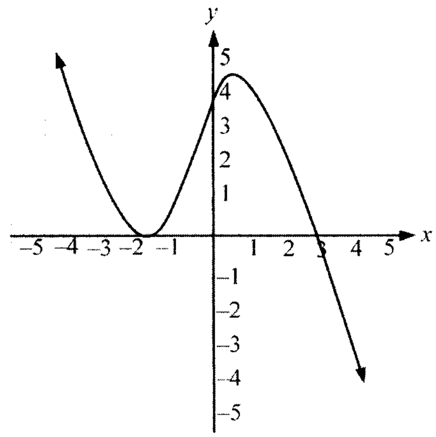


Written Response

3. Using approximate values of x, describe the concavity of the graph for all values of x.

Use the following information to answer the next question.

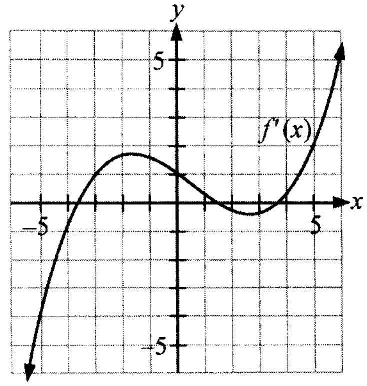
The graph of $f(x)$ is shown.



4. For which of the following approximate values of x would the graph of $f''(x)$ be equal to zero?
   A. $x = -2.3, 0.5$
   B. $x = -2.1, 3$
   C. $x = -0.6$
   D. $x = 1$

Use the following information to answer the next question.

The graph of the derivative $f'(x)$ of a smooth function $f(x)$ is shown.



5. Which of the following shapes best describes the graph of $f''(x)$ on $-1.7 < x < 2.7$?

A. Cubic

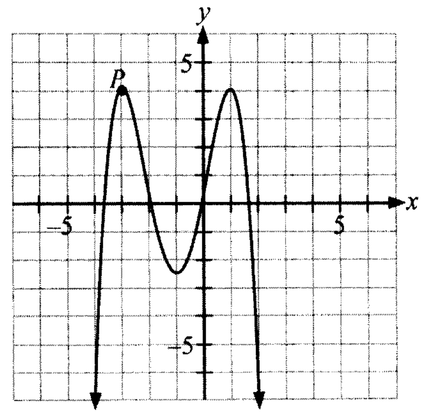
B. Parabolic

C. Diagonal line

D. Horizontal line

Use the following information to answer the next question.

The graph of a function is shown.



6. Which of the following pairs of relations describes characteristics of the first and second derivatives at point P labelled on the given graph?

A. $f'(x) = 0, f''(x) > 0$

B. $f'(x) = 0, f''(x) < 0$

C. $f'(x) > 0, f''(x) < 0$

D. $f'(x) > 0, f''(x) > 0$

7. On which of the following intervals is the graph of $f(x) = x^4 + 4x^3 - 18x^2 - 14x + 3$ concave up?

A. $-3 < x < 1$

B. $-3 < x < \infty$

C. $-\infty < x < 1$ and $-3 < x < \infty$

D. $-\infty < x < -3$ and $1 < x < \infty$

Use the following information to answer the next question.

A function is defined by
$f(x) = x^3 + ax^2 + bx + 1$.

Written Response

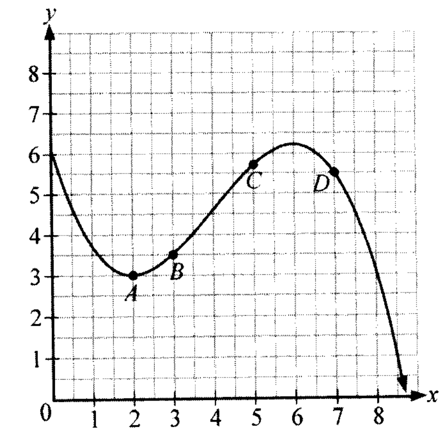
8. For what values of the constants a and b would $(1, 6)$ be a point of inflection on the graph of $f(x)$?

Written Response

9. Determine the first and second derivatives of the function $f(x) = -\frac{5}{x^3}$.

Use the following information to answer the next question.

The partial graph of a polynomial function is shown.



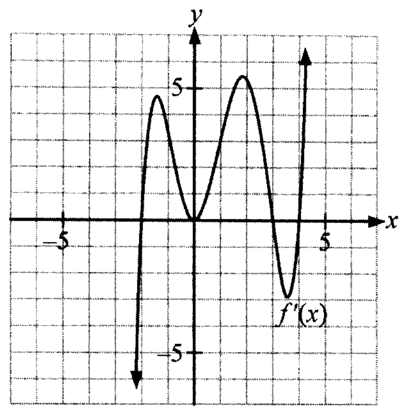
10. At which point on the graph are the first and second derivatives both positive?

A. *A* **B.** *B*

C. *C* **D.** *D*

Use the following information to answer the next question.

The graph of $f'(x)$, the derivative of a function, $f(x)$, is shown.



11. For approximately which of the following x-values does the graph of $f(x)$ have a local minimum?

A. $x = -2, 4$

B. $x = -2, 0$

C. $x = -3.5, 0, 4$

D. $x = -2, 0, 3, 4$

Use the following information to answer the next question.

A continuous function satisfies all of the following conditions:

A. $f(0) = f(3) = 0,$
$f(-1) = f(1) = -2$

B. $f'(-1) = f'(0) = f'(1) = 0$

C. $f'(x) < 0$ on
$-\infty < x < -1$ and $0 < x < 1$

D. $f'(x) > 0$ on
$-1 < x < 0$ and $1 < x < \infty$

E. $f''\left(-\frac{1}{2}\right) = f''\left(\frac{1}{2}\right) = f''(2) = 0$

F. $f''(x) > 0$ on
$-\infty < x < -\frac{1}{2}$ and $\frac{1}{2} < x < 2$

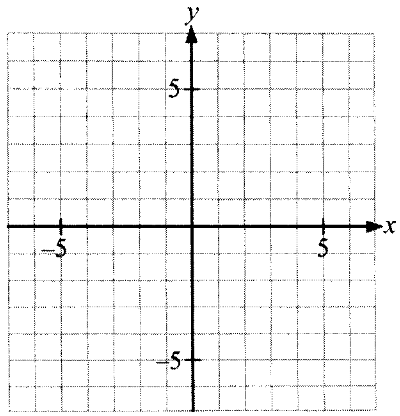
G. $f''(x) < 0$ on
$-\frac{1}{2} < x < \frac{1}{2}$ and $2 < x < \infty$

H. $\lim_{x \to \infty} f(x) = \infty$

I. $\lim_{x \to -\infty} f(x) = \infty$

Written Response

12. On the axes provided, sketch the graph of a continuous function that satisfies all of the given conditions.



13. Which of the following steps are necessary for finding the local extrema of a polynomial function, $f(x)$?

A. Finding $f''(x)$ and solving $f(x) = f'(x)$ for x

B. Finding $f'(x)$ and solving $f(x) = f'(x)$ for x

C. Finding $f''(x)$ and solving $f''(x) = 0$ for x

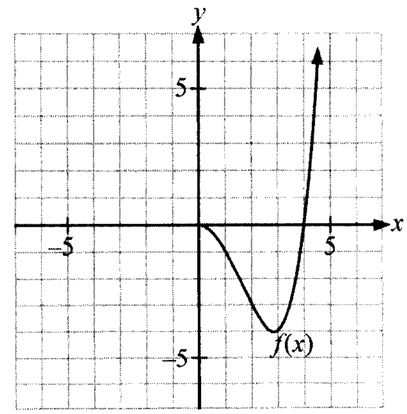
D. Finding $f'(x)$ and solving $f'(x) = 0$ for x

Numerical Response

14. The y-coordinate of the point of inflection of $f(x) = x^3 - 6x^2 + 11x + 6$ is ________.

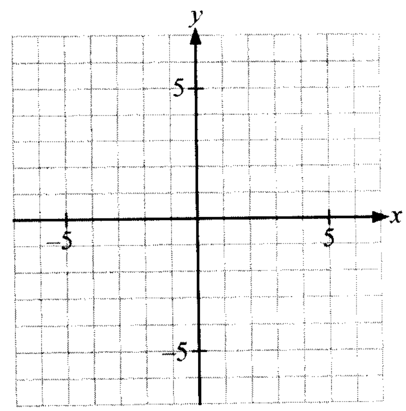
Use the following information to answer the next question.

Half the graph of an even polynomial function $f(x)$ is shown.



Written Response

15. Sketch the entire graph of $f(x)$ on the grid provided.



Use the following information to answer the next question.

The position function of a particle is given by $s(t) = 2t^3 - 24t^2 + 90t + 7$ for $t \geq 0$.

16. For what values of t is the velocity of the particle increasing?

A. $t > 4$ only

B. $t > 5$ only

C. $3 < t < 4$ only

D. $0 < t < 3$ and $t > 5$

Use the following information to answer the next question.

The displacement of an object is defined by the function $s(t) = 4t^3 - 9t^2 + 6t + 2$, $t \geq 0$, in which s is the displacement in metres, and t is the time in seconds.

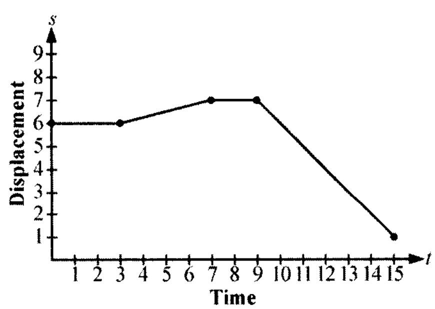
17. What is the total distance travelled by the object in the first 4 s?

A. 126.0 m **B.** 136.5 m

C. 138.0 m **D.** 146.5 m

Use the following information to answer the next question.

The displacement-time graph of an object is shown.



Numerical Response

18. The instantaneous acceleration of the object at 11 s appears to be ________ m/s^2.

Use the following information to answer the next question.

The power loss, P, in watts per kilometre in a particular electric conductor is given by $P = C^2r - \frac{A}{r}$, where C is the current in amps, A is a positive constant, and r is the resistance in ohms per kilometre (Ω/km).

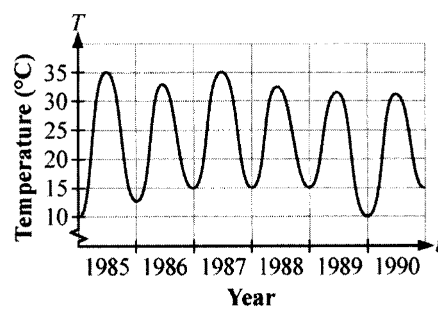
19. For a given current, what is the resistance required for a minimum loss of power per kilometre?

A. $\frac{A}{C^2}$ Ω/km **B.** $\frac{\sqrt{A}}{C}$ Ω/km

C. $\frac{\sqrt{C}}{A}$ Ω/km **D.** $\sqrt{\frac{A}{2C}}$ Ω/km

Use the following information to answer the next question.

The average monthly daytime temperatures of a city in the United States over a period of six years were recorded and connected in a smooth curve as shown in the graph.



20. During which year did the average monthly temperature have the **greatest** sustained rate of increase between January and June?

A. 1985 **B.** 1986

C. 1987 **D.** 1990

Use the following information to answer the next question.

The rate of change of the height in metres of a rocket is given by the function $v(t) = 24 + 4t^2$, where t is the time in seconds after liftoff.

Numerical Response

21. The rate of change of the height of the rocket 7 s after liftoff is ________ m/s.

Use the following information to answer the next multipart question.

22. The rate at which cars pass through an intersection is known as traffic flow. The cars were counted at a busy intersection over a 30 min period of time. The function $f(t) = 6\cos\left(\frac{t}{2} - \pi\right) + 4t$, where $0 \le t \le 30$, models the total number of cars passing through the intersection after t minutes.

a) What was the approximate traffic flow 8 min into the monitoring period?

A. 1.73 cars/min

B. 2.27 cars/min

C. 3.32 cars/min

D. 4.12 cars/min

Written Response

b) At 8 min into the monitoring period, was the traffic flow increasing or decreasing?

Use the following information to answer the next question.

The volume of gasoline, $g(x)$, in litres used by a sport utility vehicle to travel a distance, x, in kilometres is defined by the function $g(x) = 0.06x\left(2 - e^{-\frac{x}{3}}\right)$.

23. What is the rate of gasoline consumption 250 km into a trip?

A. 0.12 L/km **B.** 3.21 L/km

C. 30.00 L/km **D.** 30.12 L/km

Use the following information to answer the next question.

A small community's water tank holds 1 500 gal of water at midnight ($t = 0$). Over the course of the next 18 h, fresh water is pumped into the tower at a rate defined by $R(t) = 85\sqrt{t}\sin\left(\frac{t}{5}\right)$, where $R(t)$ is in gal/h. Over the same 18 h, water is withdrawn from the tower such that the total amount withdrawn from the tank, in gallons, after t hours is defined by $T(t) = -520\cos\left(\frac{t}{2}\right)$.

Numerical Response

24. To the nearest tenth of an hour, how long after midnight does the water tank's volume begin to increase? ________h

Use the following information to answer the next question.

When the brightness, b, of a flashlight shone into a person's eye is increased, the eye reacts by decreasing its pupil size, P. The sensitivity, S, is defined to be the rate of change of pupil size with respect to brightness. The equation $P = \dfrac{42 + 25b^{0.3}}{1 + 4b^{0.3}}$ shows the dependence of pupil size on the brightness, in which P is measured in square millimetres, and b is measured in appropriate units of brightness.

Written Response

25. To the nearest hundredth of a square millimetre per unit of brightness, using the unsimplified equation for the sensitivity, evaluate the sensitivity for a brightness of 7 units.

Use the following information to answer the next question.

After colliding with an iceberg, a ship began to take on water at a rate defined by $R(t) = 2\,000te^{-0.2t}$, where $R(t)$ is measured in litres per hour and t is the number of hours since the collision.

26. How long after the collision was the rate of water intake at a maximum?

A. 0 h **B.** 5.0 h

C. 10.0 h **D.** 15.0 h

Use the following information to answer the next question.

Graham is designing a new home and wants to put in a Norman window, which is a window in the shape of a rectangle with a semicircle on top. He wants to use the 12 m of window casing he already has in storage for the perimeter of the window. The perimeter of the window is given by the equation $P = 2x + 2r + \pi r$, in which x is the height of the rectangular portion of the window in metres, and r is the radius of the semicircle in metres.

27. If area is given by the equation $A = 2xr + \dfrac{\pi r^2}{2}$, what radius will yield a window with maximum area?

A. $\dfrac{6}{4+\pi}$ m **B.** $\dfrac{12}{4+\pi}$ m

C. $\dfrac{24}{4+\pi}$ m **D.** $\dfrac{48}{4+\pi}$ m

Use the following information to answer the next question.

Rumours in a high school spread according to the equation $s(t) = \dfrac{1}{1 + 10e^{-0.5t}}$, where $s(t)$ is the proportion of the school's population that knows the rumour at time t in minutes.

Numerical Response

28. To the nearest tenth of a minute, the rate of spread of a rumour is a maximum after __________ min.

Use the following information to answer the next question.

The function $v(t)$ defined by $v(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$ is used to model the velocity of a test plane in miles per minute for $0 \le t \le 15$.

29. Rounded to the nearest hundredth, what is the acceleration of the plane after 8 min, assuming this model is valid for all flights in the test plane?

A. 0.02 mi/min² **B.** 0.04 mi/min²

C. 0.32 mi/min² **D.** 0.34 mi/min²

Use the following information to answer the next question.

A family driving across the United States in the summer experienced a delay when their motor home's engine overheated to 200°C. After waiting 7 min in the scorching 30°C heat, they checked the engine's temperature and discovered it had cooled to 160°C. The engine must cool to 70°C before they can resume driving.

The change in temperature of the motor home's engine follows Newton's law of cooling, represented by the exponential relationship $T - T_s = (T_0 - T_s)e^{kt}$, in which T is the object's temperature at time t, T_s is the temperature of the surroundings, T_0 is the initial temperature of the object, and k is a constant representing the relative rate of cooling of the given object.

30. If the temperature difference between the engine and the air outside changes at a rate proportional to this temperature difference, then the rate of decrease of the engine's temperature at the instant when the family can get moving again, to the nearest hundredth, is

A. 37.76°C/min

B. 2.68°C/min

C. −1.53°C/min

D. −23.15°C/min

Numerical Response

31. A small city increased in population from 45 000 to 70 000 in 35 years. Rounded to the nearest whole number, what will the rate of population increase be when the city grows to 90 000, if the population grows exponentially? _______ people/yr

Use the following information to answer the next question.

The half-life of a highly reactive substance is 6 months. The law of natural decay uses a function of the form $f(t) = Ae^{kt}$ to describe exponential decay, where $f(t)$ is the amount present after time t, A is the initial amount present, and k is a constant to be determined from given information.

Written Response

32. Use the law of natural decay to determine to the nearest tenth when the rate of decrease of mass will reach 7 kg/month if there is 84 kg initially.

ANSWERS AND SOLUTIONS — UNIT TEST

1. A	7. D	14. 12	21. 220	27. B
2. a) WR	8. WR	15. WR	22. a) A	28. 4.6
b) WR	9. WR	16. A	b) WR	29. A
3. WR	10. B	17. B	23. A	30. C
4. C	11. A	18. 0	24. 5.0	31. 1136
5. B	12. WR	19. B	25. WR	32. WR
6. B	13. D	20. A	26. B	

1. A

Step 1
Analyze the first pair of graphs.
In the first pair, the slope of the first graph is constant.

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{0-(-4)}{0-(-2)} = 2$$

The graph of its derivative should be a horizontal line at $y = 2$. Therefore, the second graph is the derivative graph of the first graph.

Step 2
Analyze the second pair of graphs.
In the second pair, the first graph is a quadratic function and its derivative graph is a quadratic function. This is incorrect since the derivative graph of a quadratic function is a linear function. Therefore, the second graph is not the derivative graph of the first graph.

Step 3
Analyze the third pair of graphs.
Analyze the slope of the first graph in the third pair from left to right.

- Where $-\infty < x < -3$, the slope is negative and increasing.
- Where $x = -3$, the slope is zero.
- Where $-3 < x < 0$, the slope is positive and increasing.
- Where $0 < x < 3$, the slope is positive and decreasing.
- Where $x = 3$, the slope is zero.
- Where $3 < x < \infty$, the slope is negative and decreasing.

Graphing the slope values of $f(x)$ gives the shape of the graph of the derivative. In the second graph, the y-values start positive and decrease and then start to increase at $x = 0$. The y-values of the second graph do not correspond to the slope of the first graph. Therefore, the second graph is not the derivative graph of the first graph.

Step 4
Analyze the fourth pair of graphs.
In the fourth pair, the first graph is a cubic function and its derivative graph is a quadratic function. Inspect the points on the first graph to see that the first graph is the function $f(x) = x^3$. The derivative of $f(x)$ is $f'(x) = 3x^2$. Examine a few key points on the second graph, such as $(-1, 3)$, $(0, 0)$, and $(1, 3)$. The second graph represents $f'(x) = 3x^2$. Therefore, the second graph is the derivative graph of the first graph.
Pairs 1 and 4 represent the graph of a function and the graph of its derivative.

2. a) WR

Since the original function, $f(x)$, has a degree of 3, its derivative function, $f'(x)$, will have a degree of 2. Since $f'(x)$ has a degree of 2 and $f(x)$ has a positive leading coefficient, the graph of $f'(x)$ will be a parabola that opens upward.

b) WR

The graph of $g'(x)$ would be a reflection in the x-axis of the graph of $f'(x)$.

3. WR

The concavity of the graph for all values of x may be described as follows:

- On $x < -2$, the graph of the function is concave down.
- At $x = -2$, there is a point of inflection.
- On $-2 < x < 0$, the graph of the function is concave up.
- At $x = 0$, there is a point of inflection.
- On $0 < x < 3$, the graph of the function is concave down.
- At $x = 3$, there is a point of inflection.
- On $x > 3$, the graph of the function is concave up.

4. C

When the graph of $f''(x)$ is equal to zero, there is a point of inflection on the graph of $f(x)$. At $x = -0.6$ on the graph of $f(x)$, there is a point of inflection.

Therefore, the graph of $f''(x)$ is equal to zero at $x = -0.6$.

5. B

Using approximate x values, analyze the slope of the derivative graph on the interval $-1.7 < x < 2.7$ when it is viewed from left to right.

- $x = -1.7$, slope is zero
- $-1.7 < x < 0.8$, slope negative and decreasing
- $0.8 < x < 2.7$, slope is negative and increasing
- $x = 2.7$, slope is zero

When the approximate slope values are graphed, the result is the following:

- $x = -1.7$, there is an x intercept
- $-1.7 < x < 0.8$, y values are negative and decreasing
- $x = 0.8$, there is a local minimum
- $0.8 < x < 2.7$, y values are negative and increasing
- $x = 2.7$, there is an x intercept

Therefore, the result is a parabolic shape with a local minimum. Also, since the first derivative graph represents a cubic function, the second derivative graph should represent a quadratic function.

6. B

Point P on the graph of the function is a local maximum where the slope is zero. Therefore, the first derivative graph will be equal to zero at $x = P$. In other words, at point P on the first derivative graph, $f'(x) = 0$.

At point P on the graph of the function, the graph is concave down and decreasing. Therefore, the second derivative graph will be negative at $x = P$. In other words, at point P on the second derivative graph, $f''(x) < 0$.

7. D

Step 1
Determine the first derivative of the function $f(x)$.

$f(x) = x^4 + 4x^3 - 18x^2 - 14x + 3$
$f'(x) = 4x^3 + 12x^2 - 36x - 14$

Step 2
Determine the second derivative of the function $f(x)$.

$f'(x) = 4x^3 + 12x^2 - 36x - 14$
$f''(x) = 12x^2 + 24x - 36$

Step 3
Find the inflection points.
The points of inflection occur at the points where the second derivative is equal to 0.

$$\begin{aligned} f''(x) &= 12x^2 + 24x - 36 \\ 0 &= 12x^2 + 24x - 36 \\ &= 12(x^2 + 2x - 3) \\ &= 12(x - 1)(x + 3) \end{aligned}$$

Therefore, the inflection points occur at $x = 1$ and $x = -3$.

Step 4
Identify the intervals of concavity.
The function is concave down where the second derivative is negative and concave up where it is positive.

Interval	$f''(x)$	Positive / Negative
$-\infty < x < -3$	$f''(-4) = 60$	Positive
$-3 < x < 1$	$f''(0) = -36$	Negative
$1 < x < \infty$	$f''(2) = 60$	Positive

Therefore, the function is concave up on the intervals $-\infty < x < -3$ and $1 < x < \infty$.

8. WR

The points of inflection occur at values of x when $f''(x) = 0$.

Step 1
Determine the second derivative of $f(x)$.

$f(x) = x^3 + ax^2 + bx + 1$
$f'(x) = 3x^2 + 2ax + b$
$f''(x) = 6x + 2a$

Step 2
Determine the value of a.
Since $(1, 6)$ is a point of inflection on the graph of $f(x)$, then $f''(1) = 0$.
If $f''(1) = 0$, substitute into the second derivative equation, and solve for a.

$$\begin{aligned} f''(x) &= 6x + 2a \\ f''(1) &= 6 + 2a \\ 0 &= 6 + 2a \\ a &= -3 \end{aligned}$$

Step 3
Determine the value of b.
Since the constant b is not part of the second derivative, its value can be determined by substituting the value of $a = -3$ and the coordinates of the point $(1, 6)$ into the original function $f(x) = x^3 + ax^2 + bx + 1$.

$$f(x) = x^3 + ax^2 + bx + 1$$
$$6 = (1)^3 + (-3)(1)^2 + b(1) + 1$$
$$6 = 1 - 3 + b + 1$$
$$7 = b$$

Therefore, the constants are $a = -3$ and $b = 7$.

9. WR

Step 1
Determine the first derivative of the function.

$$f(x) = -\frac{5}{x^3}$$
$$f(x) = -5x^{-3}$$
$$f'(x) = 15x^{-4}$$
$$f'(x) = \frac{15}{x^4}$$

Step 2
Determine the second derivative of the function.

$$f'(x) = \frac{15}{x^4}$$
$$f'(x) = 15x^{-4}$$
$$f''(x) = -60x^{-5}$$
$$f''(x) = -\frac{60}{x^5}$$

10. B

The first derivative graph will be positive at points on the original function where the slope is positive. The second derivative graph will be positive at points on the original function where concavity is positive.

The first derivative is positive at point B because the slope is positive. The second derivative is positive at point B because the concavity is also positive.

Points A and D are not positive on the first derivative graph because slope is zero at point A and slope is negative at point D. The first derivative is positive at point C because the slope is positive. However, the second derivative is negative at point C because the concavity is negative.

11. A

Step 1
Determine x-values of the local maxima/minima.
The points of local maxima/minima occur when the first derivative is equal to 0.
The first derivative graph is equal to zero at $x = -2, 0, 3,$ and 4. There is a local maximum or local minimum at these points.

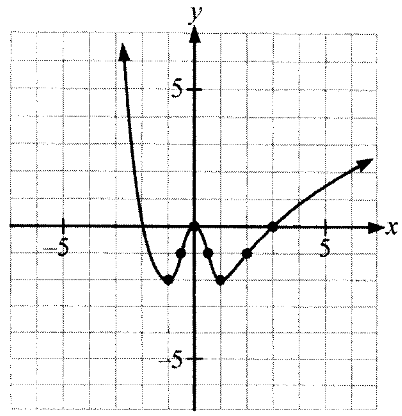
Step 2
Find the x-value(s) of the local minima.
At a local minimum point, the derivative is negative to the left and positive to the immediate right of the point.
From the derivative graph $f'(x)$, the derivative is negative to the left and positive to the immediate right of the point at $x = -2$ and $x = 4$.
Therefore, there is a local minimum at $x = -2$ and $x = 4$.

12. WR

The function graph has the following characteristics:

A. Passes through the points $(0, 0)$, $(3, 0)$, $(-1, -2)$, and $(1, -2)$.
B. There are local extrema at $(-1, -2)$, $(0, 0)$, and $(1, -2)$.
C. Decreasing on the intervals $-\infty < x < -1$ and $0 < x < 1$.
D. Increasing on the intervals $-1 < x < 0$ and $1 < x < \infty$.
E. There are points of inflection at $x = -\frac{1}{2}, \frac{1}{2}$, and 2.
F. Concave up on the intervals $-\infty < x < -\frac{1}{2}$ and $\frac{1}{2} < x < 2$.
G. Concave down on the intervals $-\frac{1}{2} < x < \frac{1}{2}$ and $2 < x < \infty$.
H. As x approaches infinity, y approaches infinity.
I. As x approaches negative infinity, y approaches infinity.

Using the characteristics listed, sketch the graph of the function.



13. D

The local extrema of a polynomial function, $f(x)$, occurs when the first derivative (slope) of the function is equal to zero. Therefore, necessary steps involved in finding the local extrema of $f(x)$ include finding the derivative function, $f'(x)$, and solving $f'(x) = 0$ for x.

14. 12

Step 1
Determine the second derivative function of $f(x)$.

$$f(x) = x^3 - 6x^2 + 11x + 6$$
$$f'(x) = 3x^2 - 12x + 11$$
$$f''(x) = 6x - 12$$

Step 2
Determine the x-coordinate of the point of inflection.
The inflection point occurs when the second derivative, $f''(x)$, is equal to zero.
Solve for x when $f''(x) = 0$.

$$f''(x) = 6x - 12$$
$$0 = 6x - 12$$
$$12 = 6x$$
$$x = 2$$

Therefore, there is an inflection point at $x = 2$.

Step 3
Determine the y-coordinate of the point of inflection.
Solve for $f(x)$ when $x = 2$.

$$f(x) = x^3 - 6x^2 + 11x + 6$$
$$f(2) = 2^3 - 6(2)^2 + 11(2) + 6$$
$$= 8 - 24 + 22 + 6$$
$$= 12$$

The y-coordinate of the point of inflection is 12.

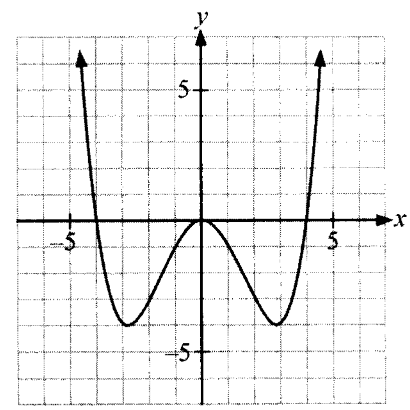
15. WR

When a polynomial is even, the graph of the polynomial is symmetrical about the y-axis. Reflect the half of graph $f(x)$ about the y-axis to obtain the other half.

Step 1
Choose key points on the half graph of $f(x)$, and multiply the x-coordinates by -1 to obtain a new set of points.

$(4.5, 5) \rightarrow (-4.5, 5)$
$(4, 0) \rightarrow (-4, 0)$
$(2.9, -4) \rightarrow (-2.9, -4)$
$(2, -3) \rightarrow (-2, -3)$
$(1, -1) \rightarrow (-1, -1)$
$(0, 0) \rightarrow (0, 0)$

Step 2
Graph the new points, and connect the points using a smooth curve.



16. A

Step 1
Determine the second derivative of $s(t)$ to get the acceleration function $a(t)$.

$$s(t) = 2t^3 - 24t^2 + 90t + 7$$
$$s'(t) = v(t) = 6t^2 - 48t + 90$$
$$s''(t) = v'(t) = a(t) = 12t - 48$$

Step 2
Determine when acceleration is zero.
Calculate t when $a(t) = 0$.

$$a(t) = 12t - 48$$
$$0 = 12t - 48$$
$$t = 4$$

Step 3
The velocity of a particle is increasing when the acceleration of the particle is positive.
Use a test point on the interval $0 < t < 4$ to determine when $v(t)$ is increasing or decreasing.
Solve $a(t)$ when $t = 1$.

$$\begin{aligned} a(t) &= 12t - 48 \\ a(1) &= 12(1) - 48 \\ &= -36 \end{aligned}$$

Acceleration is negative; therefore, the velocity of the particle is decreasing on the time interval $0 < t < 4$.

Step 4
Use a test point on the interval $4 < t < \infty$ to determine when $v(t)$ is increasing or decreasing.
Solve $a(t)$ when $t = 5$.

$$\begin{aligned} a(t) &= 12t - 48 \\ a(5) &= 12(5) - 48 \\ &= 12 \end{aligned}$$

Acceleration is positive; therefore, the velocity of the particle is increasing on the time interval $4 < t < \infty$ (i.e.,$t > 4$) only.

17. B

Step 1
Find the first derivative of the displacement function to get the velocity function.

$$s(t) = 4t^3 - 9t^2 + 6t + 2$$
$$s'(t) = v(t) = 12t^2 - 18t + 6$$

Step 2
Find when the velocity of the object is zero. That is, determine when the object stops and possibly changes direction.
Determine the value or values of t when $v(t) = 0$.

$$\begin{aligned} v(t) &= 12t^2 - 18t + 6 \\ 0 &= 6(2t^2 - 3t + 1) \\ 0 &= 6(2t^2 - 2t - 1t + 1) \\ 0 &= 6(2t(t-1) - 1(t-1)) \\ 0 &= 6(t-1)(2t-1) \\ t &= 1, t = \frac{1}{2} \end{aligned}$$

Step 3
Calculate the total distance travelled by the object.
The object is in motion over the following time intervals:

$$0 < t < \frac{1}{2}$$
$$\frac{1}{2} < t < 1$$
$$1 < t < 4$$

Take the sum of the distance travelled in each time interval to find the total distance travelled. Disregard the direction of travel in each time interval.
The total distance travelled is calculated as follows:

$$\begin{aligned} &= \left(\begin{array}{l} [s(4) - s(1)] + \left[s(1) - s\left(\frac{1}{2}\right)\right] \\ + \left[s\left(\frac{1}{2}\right) - s(0)\right] \end{array} \right) \\ &= \left(\begin{array}{l} [138 - 3] + [3 - 3.25] \\ + [3.25 - 2] \end{array} \right) \\ &= 135 + 0.25 + 1.25 \\ &= 136.5 \text{ m} \end{aligned}$$

18. 0

At $t = 11$ s, the slope of the displacement-time graph is decreasing in a linear fashion, which means that the velocity at that same time would be a negative constant. Therefore, the acceleration at $t = 11$ s would be 0 m/s^2.

19. B

Step 1
Find the derivative of the power loss function, $P = C^2r + \frac{A}{r}$, with respect to r.

$$P = C^2r + \frac{A}{r}$$
$$P = C^2r + Ar^{-1}$$
$$\frac{dP}{dr} = C^2 - Ar^{-2}$$

Step 2

Solve for r when $\frac{dP}{dr} = 0$.

$$\frac{dP}{dr} = C^2 - Ar^{-2}$$
$$0 = C^2 - \frac{A}{r^2}$$
$$\frac{A}{r^2} = C^2$$
$$\frac{A}{C^2} = r^2$$
$$\sqrt{\frac{A}{C^2}} = r$$
$$\frac{\sqrt{A}}{C} = r$$

The resistance required for a minimum loss of power per kilometre is $\frac{\sqrt{A}}{C}$ Ω/km.

20. A

In 1985, the average monthly daytime temperature for the first part of the year increased the most.

The average temperature in January 1985 started at the lowest recorded value of the given six years and, by June, had reached the highest recorded value of the given years. This resulted in the greatest six-month rate of increase.

21. 220

Since the rate of change function for the height of the rocket is given, evaluate the rate of change function (velocity function) for $t = 7$.

$$v(t) = 24 + 4t^2$$
$$v(7) = 24 + 4(7)^2 = 24 + 196 = 220$$

The rate of change of the height of the rocket 7 s after liftoff is 220 m/s.

22. a) A

Step 1

Determine the first derivative of $f(t)$.

$$f(t) = 6\cos\left(\frac{t}{2} - \pi\right) + 4t$$
$$f'(t) = -6\sin\left(\frac{t}{2} - \pi\right)\left(\frac{1}{2}\right) + 4$$
$$f'(t) = -3\sin\left(\frac{t}{2} - \pi\right) + 4$$

The derivative, $f'(t)$, models the rate of change of the number of cars (traffic flow) passing through the intersection at some time, t.

Step 2

Solve $f'(8)$ when $t = 8$. Remember to make sure the calculator is in radian mode.

$$f'(t) = -3\sin\left(\frac{t}{2} - \pi\right) + 4$$
$$f'(8) = -3\sin\left(\frac{8}{2} - \pi\right) + 4$$
$$f'(8) \approx 1.73$$

Therefore, at 8 min into the monitoring period, the traffic flow was 1.73 cars/min.

b) WR

Step 1

Determine the function that represents the rate of change of traffic flow.

Find the second derivative of $f(t)$.

$$f(t) = 6\cos\left(\frac{t}{2} - \pi\right) + 4t$$
$$f'(t) = -3\sin\left(\frac{t}{2} - \pi\right) + 4$$
$$f''(t) = -3\cos\left(\frac{t}{2} - \pi\right)\left(\frac{1}{2}\right) = -\frac{3}{2}\cos\left(\frac{t}{2} - \pi\right)$$

Step 2

Using the second derivative function, determine if traffic flow was increasing or decreasing 8 min into the monitoring period.

Solve $f''(t)$ when $t = 8$.

$$f''(t) = -\frac{3}{2}\cos\left(\frac{t}{2} - \pi\right)$$
$$f''(8) = -\frac{3}{2}\cos\left(\frac{8}{2} - \pi\right) \approx -0.98$$

Since the rate of change of the traffic flow is negative, the traffic flow was decreasing 8 min after monitoring.

23. A

Step 1

Determine the function that represents the rate of gas consumption.

Find the first derivative function of $g(x)$.

$$g(x) = 0.06x\left(2 - e^{-\frac{x}{3}}\right)$$
$$g'(x) = \left(0.06x\left(-e^{-\frac{x}{3}}\right)\left(-\frac{1}{3}\right) + \left(2 - e^{-\frac{x}{3}}\right)(0.06)\right)$$
$$= 0.02xe^{-\frac{x}{3}} + 0.12 - 0.06e^{-\frac{x}{3}}$$

Step 2
Determine the rate of gas consumption 250 km into a trip.
Solve for $g'(250)$.

$$g'(x) = \begin{pmatrix} 0.02xe^{-\frac{x}{3}} + 0.12 \\ -0.06e^{-\frac{x}{3}} \end{pmatrix}$$

$$g'(250) = \begin{pmatrix} 0.02(250)e^{-\frac{250}{3}} + 0.12 \\ -0.06e^{-\frac{250}{3}} \end{pmatrix}$$
$$= 0.12$$

The rate of gasoline consumption 250 km into a trip is 0.12 L/km.

24. 5.0

Step 1
Determine the function that represents the rate at which water is withdrawn from the water tank.
Find the first derivative of the function $T(t) = -520\cos\left(\frac{t}{2}\right)$.

$$T(t) = -520\cos\left(\frac{t}{2}\right)$$
$$T'(t) = 520\sin\left(\frac{t}{2}\right)\cdot\left(\frac{1}{2}\right)$$
$$= 260\sin\left(\frac{t}{2}\right)$$

Step 2
Determine the function that represents the rate at which the amount of water in the water tank is changing.
The rate at which the amount of water in the tank is changing, C, can be determined from the difference between the rate at which fresh water is being pumped into the tank, $R(t)$, and the rate at which water is being withdrawn, $T'(t)$.

$$C(t) = R(t) - T'(t)$$
$$= 85\sqrt{t}\sin\left(\frac{t}{5}\right) - 260\sin\left(\frac{t}{2}\right)$$

Step 3
Determine how long after midnight the water tank's volume begins to increase.
Graph the rate of change function $C(t) = 85\sqrt{t}\sin\left(\frac{t}{5}\right) - 260\sin\left(\frac{t}{2}\right)$, and determine at what time, t, it first becomes positive; this will correspond to the time at which the overall volume in the tank starts increasing.
Using a graphing calculator, enter $Y_1 = 85\sqrt{(X)}\sin(X/5) - 260\sin(X/2)$ in the [Y =] window. Then, press [WINDOW], enter x:[0, 18, 2] and y:[−1 000, 1 000, 200], and press [GRAPH].
Press [2nd] [TRACE] and select 2:zero to determine that the rate of change of the water volume is 0 when $x = t \approx 4.969$. Immediately after this time, the rate of change of water volume is positive and increasing. Therefore, the volume in the tank begins increasing 5.0 h after midnight.

25. WR

Step 1
Determine the function that represents sensitivity, S.
Since sensitivity, S, is the rate of change of pupil size, P, with respect to brightness, b, an equation for the sensitivity can be found by differentiating the given function, P, with respect to b.

$$P = \left(42 + 25b^{0.3}\right)\left(1 + 4b^{0.3}\right)^{-1}$$
$$\frac{dP}{db} = S$$
$$= \begin{pmatrix} \left(42 + 25b^{0.3}\right) \\ \times\ (-1)\left(1 + 4b^{0.3}\right)^{-2}\left(1.2b^{-0.7}\right) \\ +\ \left(1 + 4b^{0.3}\right)^{-1}\left(7.5b^{-0.7}\right) \end{pmatrix}$$

Step 2
Determine the sensitivity for a brightness of 7 units.
Calculate S when $b = 7$.

$$S = \begin{pmatrix} \left(42 + 25b^{0.3}\right) \\ \times\ (-1)\left(1 + 4b^{0.3}\right)^{-2}\left(1.2b^{-0.7}\right) \\ +\ \left(1 + 4b^{0.3}\right)^{-1}\left(7.5b^{-0.7}\right) \end{pmatrix}$$

$$S(7) = \begin{pmatrix} \left(42 + 25(7)^{0.3}\right) \\ \times\ (-1)\left(1 + 4(7)^{0.3}\right)^{-2} \times 1.2(7)^{-0.7} \\ +\ \left(1 + 4(7)^{0.3}\right)^{-1}\left(7.5(7)^{-0.7}\right) \end{pmatrix}$$
$$\approx -0.16$$

Therefore, the sensitivity for a brightness of 7 units is approximately -0.16 mm²/unit of brightness.

26. B

Step 1
Find the derivative of the given function *R(t)*.

$$R(t) = 2\,000te^{-0.2t}$$
$$R'(t) = 2\,000t(e^{-0.2t})(-0.2) + (e^{-0.2t})(2\,000)$$
$$= 2\,000e^{-0.2t}(-0.2t + 1)$$

Step 2
Determine how long it will take for the water intake to reach a maximum.

Set $R'(t) = 0$, and solve for t.

$$R'(t) = 2\,000e^{-0.2t}(-0.2t + 1)$$
$$0 = 2\,000e^{-0.2t}(-0.2t + 1)$$
$$0 = e^{-0.2t} \text{ or } 0 = -0.2t + 1$$

Since the expression $e^{-0.2t}$ will never equal 0 (it will always be positive), there is only one value of t to consider.

$$-0.2t + 1 = 0$$
$$-0.2t = -1$$
$$t = 5$$

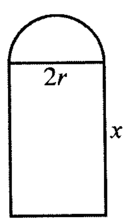
Step 3
Confirm the existence of a maximum at $t = 5$.

Interval	**$R'(t)$**	**$R(t)$**
$0 < t < 5$	+	Increasing
$5 < t < \infty$	–	Decreasing

The ship's water intake was at a maximum 5 h after the collision.

27. B

Draw a diagram of the window.



It is given that the perimeter of the window is $P = 2x + 2r + \pi r$. Since there is 12 m of window casing available, $12 = 2x + 2r + \pi r$.

Isolate x in the perimeter equation.

$$12 = 2x + 2r + \pi r$$
$$12 - 2r - \pi r = 2x$$
$$\frac{12 - 2r - \pi r}{2} = x$$

Substitute the expression of x into the area formula.

$$A = 2xr + \frac{\pi r^2}{2}$$
$$= 2\left(\frac{12 - 2r - \pi r}{2}\right)r + \frac{\pi r^2}{2}$$
$$= 12r - 2r^2 - \pi r^2 + \frac{\pi}{2}r^2$$
$$= 12r - 2r^2 - \frac{\pi}{2}r^2$$

A maximum of a function occurs when its derivative is equal to zero and the function is increasing on the left and decreasing on the right of the zero.

Determine the derivative function of A, and solve for x when $A' = 0$.

$$A = 12r - 2r^2 - \frac{\pi}{2}r^2$$
$$A' = 12 - 4r - \pi r$$
$$0 = 12 - 4r - \pi r$$
$$4r + \pi r = 12$$
$$r(4 + \pi) = 12$$
$$r = \frac{12}{(4 + \pi)}$$

Confirm the maximum as follows:

Interval	**A'**	**A**
$0 < x < \frac{12}{(4+\pi)}$	Positive	Increasing
$\frac{12}{(4+\pi)} < x < \frac{24}{(4+\pi)}$	Negative	Decreasing

When the radius, r, is $\frac{12}{(4+\pi)}$ m, the area of the window will be at a maximum.

28. 4.6

Step 1
Determine the first derivative $s'(t)$ of the function $s(t)$.

$$s(t) = \frac{1}{1 + 10e^{-0.5t}}$$
$$= (1 + 10e^{-0.5t})^{-1}$$
$$s'(t) = -1(1 + 10e^{-0.5t})^{-2}(10e^{-0.5t})(-0.5)$$
$$= \frac{5e^{-0.5t}}{(1 + 10e^{-0.5t})^2}$$

Step 2
Use technology to find the location of the derivative function's maximum.

Using a graphing calculator, press [Y =] and enter Y_1 = 5e^(−0.5X)/(1 + 10e^(−0.5X))^2.

Then, press [WINDOW], and enter x:[−10, 10, 1], y:[−0.5, 0.5, 0.1].
To find the maximum value of the derivative function, press [2nd] [TRACE], and select 4:maximum. Move the cursor to just left of the maximum, and press [ENTER]. Then, move the cursor to just right of the maximum, and press [ENTER] twice. The maximum occurs at approximately (4.6, 0.125).
Therefore, the rate of spread of a rumour is at a maximum after 4.6 min.

29. A

Since the function representing the velocity of the airplane is given, the function representing its acceleration can be determined.

Step 1
Find the derivative of the velocity function.

$$v(t) = 6 + \cos\left(\frac{t}{10}\right) + 3\sin\left(\frac{7t}{40}\right)$$
$$\begin{aligned} v'(t) &= a(t) \\ &= -\sin\left(\frac{t}{10}\right)\left(\frac{1}{10}\right) + 3\cos\left(\frac{7t}{40}\right)\left(\frac{7}{40}\right) \\ &= -\frac{1}{10}\sin\left(\frac{t}{10}\right) + \frac{21}{40}\cos\left(\frac{7t}{40}\right) \end{aligned}$$

Step 2
Determine the acceleration of the plane after 8 min. Calculate the acceleration function for $t = 8$. Remember to make sure the calculator is in radian mode.

$$\begin{aligned} a(t) &= -\frac{1}{10}\sin\left(\frac{t}{10}\right) + \frac{21}{40}\cos\left(\frac{7t}{40}\right) \\ a(8) &= -\frac{1}{10}\sin\left(\frac{8}{10}\right) + \frac{21}{40}\cos\left(\frac{7(8)}{40}\right) \\ &\approx 0.02 \end{aligned}$$

Rounded to the nearest hundredth, the acceleration of the plane after 8 min is 0.02 mi/min^2.

30. C

Step 1
Determine the exponential function that represents the motor home's engine temperature after time t. Substitute known values into the formula for Newton's law of cooling.

$$\begin{aligned} T - T_s &= (T_0 - T_s)e^{kt} \\ 160 - 30 &= (200 - 30)e^{7k} \\ 130 &= 170e^{7k} \end{aligned}$$

Solve for the constant, k.

$$\begin{aligned} 130 &= 170e^{7k} \\ \frac{130}{170} &= e^{7k} \\ \ln\left(\frac{13}{17}\right) &= 7k \\ \frac{1}{7}\ln\left(\frac{13}{17}\right) &= k \end{aligned}$$

Therefore, at time t, the engine's temperature can be determined as follows:

$$\begin{aligned} T - 30 &= (200 - 30)e^{kt} \\ T &= 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} + 30 \end{aligned}$$

Step 2
Determine the time, t, when $T = 70°\text{C}$.

$$\begin{aligned} T &= 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} + 30 \\ 70 &= 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} + 30 \\ 40 &= 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} \\ \frac{4}{17} &= e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} \\ \ln\left(\frac{4}{17}\right) &= \ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right) \\ \frac{\ln\left(\frac{4}{17}\right)}{\ln\left(\frac{13}{17}\right)} &= \frac{t}{7} \\ \frac{7\ln\left(\frac{4}{17}\right)}{\ln\left(\frac{13}{17}\right)} &= t \\ 37.755 &\approx t \end{aligned}$$

Step 3
Find the rate of change of the motor home's engine temperature at $t = 37.755$.
Take the derivative of the function
$T = 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} + 30$, and evaluate T' when $t = 37.755$.

$$T = 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} + 30$$
$$T' = 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{t}{7}\right)} \times \left(\frac{1}{7}\right)\ln\left(\frac{13}{17}\right)$$
$$T'(37.755) = 170e^{\ln\left(\frac{13}{17}\right)\left(\frac{37.755}{7}\right)} \times \left(\frac{1}{7}\right)\ln\left(\frac{13}{17}\right) \approx -1.53$$

The rate of change of the engine's temperature is approximately −1.53°C/min when the temperature of the engine reaches 70°C.

31. 1136

Step 1
Determine an exponential function that models the population's growth.

Use a function of the form $f(t) = Ae^{kt}$ to describe exponential growth, where $f(t)$ is the population after time t, A is the initial population, and k is a constant to be determined from the given information.
Solve for A.
At $t = 0$, the population is 45 000.

$$f(0) = Ae^{0k}$$
$$45\,000 = A$$

After 35 years, the population of the city has grown to 70 000. Therefore, $f(35) = 70\,000$.
Solve for k.

$$f(t) = 45\,000e^{kt}$$
$$70\,000 = 45\,000e^{k(35)}$$
$$\frac{14}{9} = e^{35k}$$
$$\ln\left(\frac{14}{9}\right) = \ln e^{35k}$$
$$\ln\left(\frac{14}{9}\right) = 35k$$
$$\frac{1}{35}\ln\left(\frac{14}{9}\right) = k$$

Therefore, the exponential function is
$f(t) = 45\,000e^{\ln\left(\frac{14}{9}\right)\left(\frac{t}{35}\right)}$.

Step 2
Determine when the population will reach 90 000.
Set the function $f(t) = 90\,000$, and solve for t.

$$f(t) = 45\,000e^{\ln\left(\frac{14}{9}\right)\left(\frac{t}{35}\right)}$$
$$90\,000 = 45\,000e^{\ln\left(\frac{14}{9}\right)\left(\frac{t}{35}\right)}$$
$$2 = e^{\ln\left(\frac{14}{9}\right)\left(\frac{t}{35}\right)}$$
$$\ln 2 = \ln\left(\frac{14}{9}\right)\left(\frac{t}{35}\right)$$
$$\frac{\ln 2}{\ln\left(\frac{14}{9}\right)} = \frac{t}{35}$$
$$\frac{35\ln 2}{\ln\left(\frac{14}{9}\right)} = t$$
$$54.908 \approx t$$

The population will reach 90 000 after approximately 54.908 years.

Step 3
To determine the rate of population increase when the population reaches 90 000, determine the derivative, $f'(t)$, and evaluate for $t = 54.908$.

$$f(t) = 45\,000e^{\ln\left(\frac{14}{9}\right)\left(\frac{t}{35}\right)}$$
$$f'(t) = \left(45\,000e^{\ln\left(\frac{14}{9}\right)\left(\frac{t}{35}\right)} \times \left(\frac{1}{35}\right)\ln\left(\frac{14}{9}\right)\right)$$
$$f'(54.908) = \left(45\,000e^{\ln\left(\frac{14}{9}\right)\left(\frac{54.908}{35}\right)} \times \left(\frac{1}{35}\right)\ln\left(\frac{14}{9}\right)\right) \approx 1\,136$$

When the population reaches 90 000 in 54.908 years, the rate of population increase will be approximately 1 136 people/yr.

32. WR

Step 1
Determine an exponential function that models the decay of the reactive substance.
When the time is equal to zero, there is 84 kg of the substance.
Solve for A.

$$f(t) = Ae^{kt}$$
$$f(0) = Ae^{0k}$$
$$84 = A$$

The half-life of the substance is 6 months.
This means it takes 6 months for the substance to decrease to half its initial amount. Therefore, $f(6) = 42$.
Solve for k.

$$f(t) = 84e^{kt}$$
$$42 = 84e^{k(6)}$$
$$\frac{1}{2} = e^{6k}$$
$$\ln\left(\frac{1}{2}\right) = 6k$$
$$\frac{1}{6}\ln\left(2^{-1}\right) = k$$
$$-\frac{1}{6}\ln 2 = k$$

Therefore, the exponential function to work with is $f(t) = 84e^{-\ln 2\left(\frac{t}{6}\right)}$.

Step 2
Find the first derivative of the function $f(t)$.

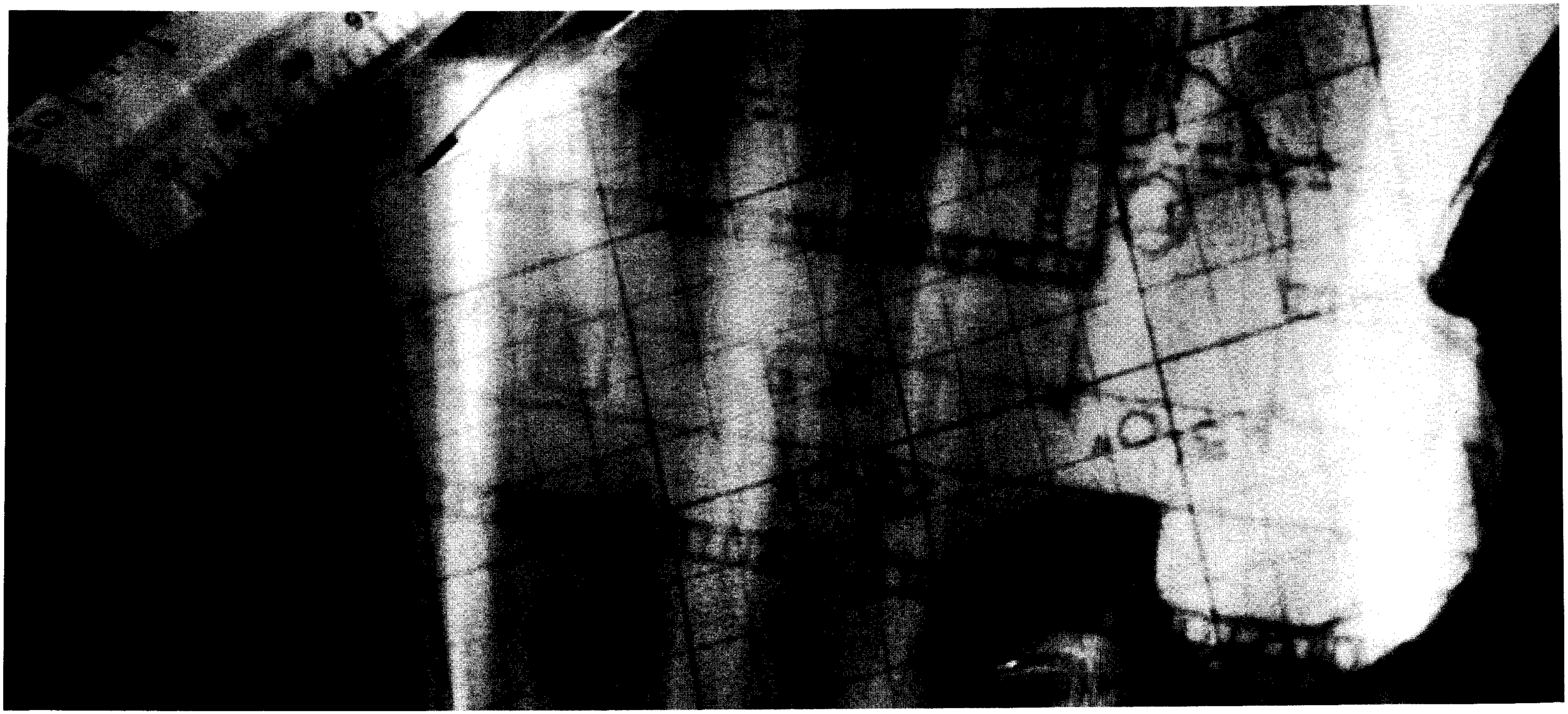
$$f(t) = 84e^{-\ln 2\left(\frac{t}{6}\right)}$$
$$f'(t) = 84e^{-\ln 2\left(\frac{t}{6}\right)}\left(-\frac{1}{6}\ln 2\right)$$

Solve for t when $f'(t) = -7$.

$$f'(t) = 84e^{-\ln 2\left(\frac{t}{6}\right)}\left(-\frac{1}{6}\ln 2\right)$$
$$-7 = 84e^{-\ln 2\left(\frac{t}{6}\right)}\left(-\frac{1}{6}\ln 2\right)$$
$$\frac{-7}{84\left(-\frac{1}{6}\ln 2\right)} = e^{-\ln 2\left(\frac{t}{6}\right)}$$
$$\frac{-7}{-14\ln 2} = e^{-\ln 2\left(\frac{t}{6}\right)}$$
$$\frac{1}{2\ln 2} = e^{-\ln 2\left(\frac{t}{6}\right)}$$
$$\ln\left(\frac{1}{2\ln 2}\right) = -\ln 2\left(\frac{t}{6}\right)$$
$$\frac{\ln\left(\frac{1}{2\ln 2}\right)}{-\ln 2} = \frac{t}{6}$$
$$\frac{6\ln\left(\frac{1}{2\ln 2}\right)}{-\ln 2} = t$$
$$2.8 \approx t$$

The rate of decrease in mass will reach 7 kg/month after approximately 2.8 months.

Geometry and Algebra of Vectors



GEOMETRY AND ALGEBRA OF VECTORS

Table of Correlations

	Outcome	Practice Questions	Unit Test Questions	Practice Test 1	Practice Test 2
VEC1.0	Representing Vectors Geometrically and Algebraically				
VEC1.1	*recognize a vector as a quantity with both magnitude and direction, and identify, gather, and interpret information about real-world applications of vectors (e.g., displacement, forces involved in structural design, simple animation of computer graphics, velocity determined using GPS)*	1, 2	1, 2		29
VEC1.2	*represent a vector in two-space geometrically as a directed line segment, with directions expressed in different ways (e.g., 320°; N 40° W), and algebraically (e.g., using Cartesian coordinates; using polar coordinates), and recognize vectors with the same magnitude and direction but different positions as equal vectors*	3, 4	3	31	
VEC1.3	*determine, using trigonometric relationships, the Cartesian representation of a vector in two-space given as a directed line segment, or the representation as a directed line segment of a vector in two-space given in Cartesian form x*	5	4	32	30
VEC1.4	*recognize that points and vectors in three-space can both be represented using Cartesian coordinates, and determine the distance between two points and the magnitude of a vector using their Cartesian representations*	6, 7	5, 6		
VEC2.0	Operating With Vectors				
VEC2.1	*perform the operations of addition, subtraction, and scalar multiplication on vectors represented as directed line segments in two-space, and on vectors represented in Cartesian form in two-space and three-space*	8, 9, 10	7, 8, 9	33	31
VEC2.2	*determine, through investigation with and without technology, some properties (e.g., commutative, associative, and distributive properties) of the operations of addition, subtraction, and scalar multiplication of vectors*	11			
VEC2.3	*solve problems involving the addition, subtraction, and scalar multiplication of vectors, including problems arising from real-world applications*	12a, 12b, 13a, 13b	10a, 10b, 11	34a, 34b	32
VEC2.4	*perform the operation of dot product on two vectors represented as directed line segments (i.e., using vector a – vector b = vector ab cosine θ) and in Cartesian form (i.e., using vector a – vector b = a^1b^1 plus a^2b^2 or vector a minus vector b equals a^1b^1 plus a^2b^2 plus a^3b^3) in two-space and three-space, and describe applications of the dot product (e.g., determining the angle between two vectors; determining the projection of one vector onto another)*	14, 15	12, 13		33
VEC2.5	*determine, through investigation, properties of the dot product (e.g., investigate whether it is commutative, distributive, or associative; investigate the dot product of a vector with itself and the dot product of orthogonal vectors)*		14		

VEC2.6	*perform the operation of cross product on two vectors represented in Cartesian form in three-space [i.e., using vector a x vector b = $(a^2b^3 - a^3b^2, a^3b^1 - a^1b^3, a^1b^2 - a^2b^1)$], determine the magnitude of the cross product (i.e., using | vector a x vector b | = | vector a | | vector b | sin θ), and describe applications of the cross product (e.g., determining a vector orthogonal to two given vectors; determining the turning effect [or torque] when a force is applied to a wrench at different angles)*	18, 19	16a, 16b, 16c	36, 37	35
VEC2.7	*determine, through investigation, properties of the cross product (e.g., investigate whether it is commutative, distributive, or associative; investigate the cross product of collinear vectors)*	20	17		
VEC2.8	*solve problems involving dot product and cross product (e.g., determining projections, the area of a parallelogram, the volume of a parallelepiped), including problems arising from real-world applications (e.g., determining work, torque, ground speed, velocity, force)*	16a, 16b, 16c, 17, 21	15, 18, 19	35, 38	34, 36a, 36b
VEC3.0	Describing Lines and Planes Using Linear Equations				
VEC3.1	*recognize that the solution points (x, y) in two-space of a single linear equation in two variables form a line and that the solution points (x, y) in two-space of a system of two linear equations in two variables determine the point of intersection of two lines, if the lines are not coincident or parallel*	22	20	39	37
VEC3.2	*determine, through investigation with technology (i.e., 3-D graphing software) and without technology, that the solution points (x, y, z) in three-space of a single linear equation in three variables form a plane and that the solution points (x, y, z) in three-space of a system of two linear equations in three variables form the line of intersection of two planes, if the planes are not coincident or parallel*	23, 24	21, 22	40	38
VEC3.3	*determine, through investigation using a variety of tools and strategies (e.g., modelling with cardboard sheets and drinking straws; sketching on isometric graph paper), different geometric configurations of combinations of up to three lines and/or planes in three-space (e.g., two skew lines, three parallel planes, two intersecting planes, an intersecting line and plane); organize the configurations based on whether they intersect and, if so, how they intersect (i.e., in a point, in a line, in a plane)*	25	37	47	
VEC4.0	Describing Lines and Planes Using Scalar, Vector, and Parametric Equations				
VEC4.1	*recognize a scalar equation for a line in two-space to be an equation of the form Ax + By + C = 0, represent a line in two-space using a vector equation (i.e., vector r = vector r 0 + t vector m.) and parametric equations, and make connections between a scalar equation, a vector equation, and parametric equations of a line in two-space*	26	23, 24	41a, 41b	39
VEC4.2	*recognize that a line in three-space cannot be represented by a scalar equation, and represent a line in three-space using the scalar equations of two intersecting planes and using vector and parametric equations (e.g., given a direction vector and a point on the line, or given two points on the line)*	27, 28	25, 26	42	40
VEC4.3	*recognize a normal to a plane geometrically (i.e., as a vector perpendicular to the plane) and algebraically, and determine, through investigation, some geometric properties of the plane (e.g., the direction of any normal to a plane is constant; all scalar multiples of a normal to a plane are also normals to that plane; three non-collinear points determine a plane; the resultant, or sum, of any two vectors in a plane also lies in the plane)*	29, 30a, 30b	27, 28	43	

VEC4.4	*recognize a scalar equation for a plane in three-space to be an equation of the form $Ax + By + Cz + D = 0$ whose solution points make up the plane, determine the intersection of three planes represented using scalar equations by solving a system of three linear equations in three unknowns algebraically (e.g., by using elimination or substitution), and make connections between the algebraic solution and the geometric configuration of the three planes*	31, 32	29, 30		41
VEC4.5	*determine, using properties of a plane, the scalar, vector, and parametric equations of a plane*	33, 34	31, 32	44	42
VEC4.6	*determine the equation of a plane in its scalar, vector, or parametric form, given another of these forms*	35, 36	33, 34	45	43
VEC4.7	*solve problems relating to lines and planes in three-space that are represented in a variety of ways (e.g., scalar, vector, parametric equations) and involving distances (e.g., between a point and a plane; between two skew lines) or intersections (e.g., of two lines, of a line and a plane), and interpret the result geometrically*	37, 38	35, 36a, 36b	46	44

VEC1.1 recognize a vector as a quantity with both magnitude and direction, and identify, gather, and interpret information about real-world applications of vectors (e.g., displacement, forces involved in structural design, simple animation of computer graphics, velocity determined using GPS)

VEC1.2 represent a vector in two-space geometrically as a directed line segment, with directions expressed in different ways (e.g., 320°; N 40° W), and algebraically (e.g., using Cartesian coordinates; using polar coordinates), and recognize vectors with the same magnitude and direction but different positions as equal vectors

Representing Vectors in Two-Dimensional Space

Quantities that have only magnitude are called scalar quantities. The magnitude of a scalar quantity can be specified by a real number. Mass, speed, distance, volume, and time are all examples of scalar quantities.

Quantities that have both magnitude and direction are called vector quantities. Vector quantities are specified by a real number for their magnitude and by a direction in space. Force, displacement, velocity, and acceleration are examples of vector quantities.

Geometric Vectors

A geometric vector is represented in two-dimensional space, or two space, by a directed line segment that is drawn as an arrow. The length of the line segment represents the magnitude, and the arrowhead points in the direction of the vector.

The direction of a geometric vector in two-dimensional space can be expressed in several different ways:

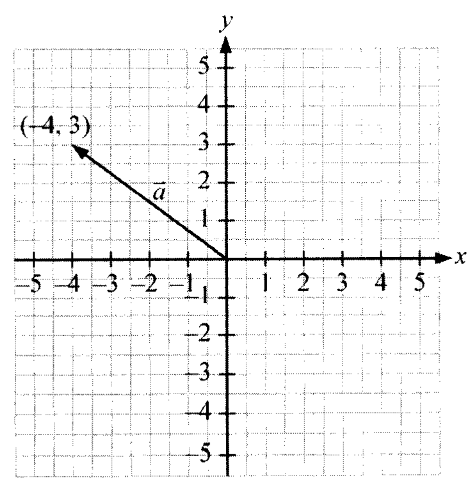
- It can be expressed as an angle measure relative to the positive horizontal axis on the Cartesian plane. A rotation counterclockwise is positive; a rotation clockwise is negative.
- It can be expressed as a value using compass directions and angle measurements. The directions of north or south are listed first, with the angle toward east or west listed second.
- It can be expressed as a bearing. The direction is measured in degrees clockwise from north so that due north is a bearing of 0°, east is a bearing of 90°, and so on.

Algebraic Vectors

Vectors in two-dimensional space are often represented algebraically using either Cartesian coordinates or polar coordinates.

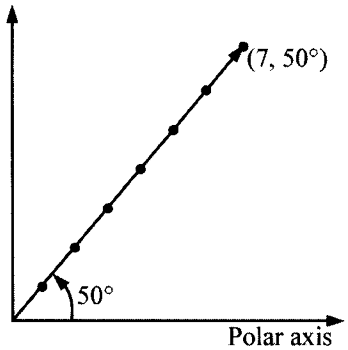
Cartesian coordinates are used if a vector is positioned on the Cartesian plane with its tail at the origin. The ordered pair of coordinates at the end (head) of the vector can represent the vector.

For example, vector $\vec{a}$ can be written algebraically as $\vec{a} = (-4, 3)$.



Polar coordinates are used if the position of a vector on a plane can be described by its magnitude (length) and the angle it makes with a horizontal polar axis. The counterclockwise rotation from the horizontal axis is positive, and the clockwise rotation is negative.

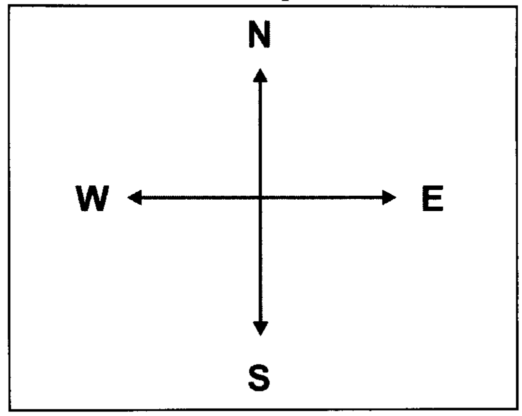
For example, a vector with polar coordinates (7, 50°) can be drawn.



1. Which of the following measurements is considered a vector quantity?
   A. Mass **B.** Speed
   C. Weight **D.** Volume

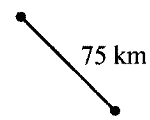
2. Which of the following quantities is a scalar quantity?
   A. Displacement **B.** Temperature
   C. Velocity **D.** Force

Use the following information to answer the next question.

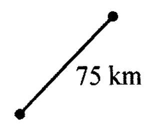


3. Using the given image as a reference for the direction of north, which of the following diagrams shows a geometric vector representing a displacement of 75 km northwest?

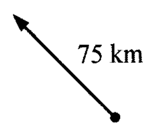
   A.



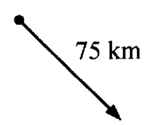
   B.



   C.



   D.



4. An algebraic vector with Cartesian coordinates of (0, 4) has polar coordinates of
   A. (4, 0°) **B.** (4, 30°)
   C. (0, 90°) **D.** (4, 90°)

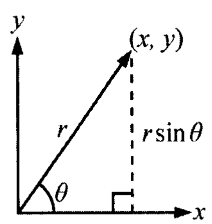
VEC1.3 determine, using trigonometric relationships, the Cartesian representation of a vector in two-space given as a directed line segment, or the representation as a directed line segment of a vector in two-space given in Cartesian form x

Represent Vectors Using Cartesian and Polar Coordinates

Consider an arbitrary vector $\overrightarrow{AB}$ in two space defined by the points $A(x_1, y_1)$ and $B(x_2, y_2)$. The Cartesian representation of $\overrightarrow{AB}$ can be determined by subtracting the coordinates of the tail, A, from the coordinates of the head, B:
$\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$.

The magnitude of $\overrightarrow{AB}$ can be calculated using the Pythagorean theorem.
$\overrightarrow{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

The relationship between Cartesian coordinates (x, y) and polar coordinates (r, θ) is shown in the given diagram.



The value of r can be determined using the Pythagorean theorem.
$r = \sqrt{x^2 + y^2}$

Since $\cos\theta = \frac{x}{r}$, then $x = r\cos\theta$.

Since $\sin\theta = \frac{y}{r}$, then $y = r\sin\theta$.

Also, $\tan\theta = \frac{y}{x}$.

These relationships can be used to change coordinates from polar form to Cartesian form and vice versa.

5. What is the approximate Cartesian form of a vector with a magnitude of 16 and a rotational direction of 65° counterclockwise to the positive x-axis?

A. (7.23, 15.00)
B. (6.76, 14.50)
C. (5.25, 12.33)
D. (8.91, 13.74)

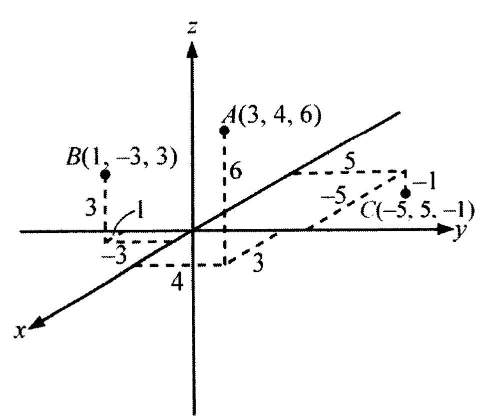
VEC1.4 recognize that points and vectors in three-space can both be represented using Cartesian coordinates, and determine the distance between two points and the magnitude of a vector using their Cartesian representations

Representing Vectors and Their Magnitudes in Three Space

Vectors are usually treated as two-dimensional objects in two space and are denoted by $\mathbb{R}^2$. However, vectors can also be treated as three-dimensional objects. Three-dimensional vectors in three space are denoted by $\mathbb{R}^3$. Points in three space are represented by ordered triplets of real numbers based on three axes that are perpendicular to each other.

The axes are labelled using x, y, and z so that any point has the form (x, y, z).

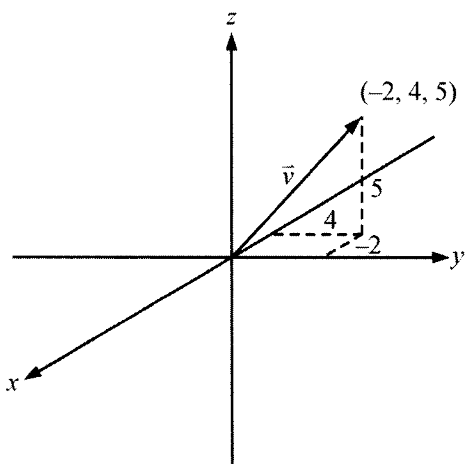
A representation of the three axes with the points $A(3, 4, 6)$, $B(1, -3, 3)$, and $C(-5, 5, -1)$ is given.



The distance between two points in three space, $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$, is given by
$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

Algebraic vectors in three space are represented using the Cartesian coordinates of the point. The coordinates (x, y, z) are located at the end of a directed line segment that starts at the origin.

The vector $\vec{v} = (-2, 4, 5)$ is the vector that starts at the origin and terminates at $(-2, 4, 5)$, as shown.



The magnitude of the algebraic vector $\vec{v} = (x, y, z)$ is found by using the distance formula and the points $(0, 0, 0)$ and (x, y, z) so that $|\vec{v}| = \sqrt{x^2 + y^2 + z^2}$.

6. What is the distance between the points $A(4, 3, -7)$ and $B(-2, 4, 2)$?
   A. $\sqrt{55}$ **B.** $\sqrt{118}$
   C. $3\sqrt{14}$ **D.** $6\sqrt{62}$

7. Calculated to the nearest hundredth, what is the magnitude of $v = (3, 7, -2)$?
   A. 6.83 **B.** 7.35
   C. 7.87 **D.** 8.04

VEC2.1 perform the operations of addition, subtraction, and scalar multiplication on vectors represented as directed line segments in two-space, and on vectors represented in Cartesian form in two-space and three-space

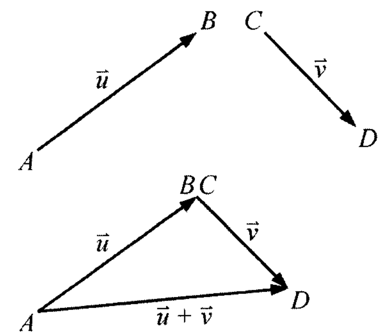
Operations with Vectors

Both geometric and algebraic vectors can undergo the common mathematical operations of addition, subtraction, and multiplication.

Adding Geometric Vectors

To add geometric vectors in two space, place the vectors so that the terminal point of one line segment is the initial point of the other line segment. The line segment from the initial point of the first vector to the terminal point of the second is the sum of the two vectors.

In the given diagram, $\overrightarrow{AB} + \overrightarrow{CD} = \overrightarrow{AD}$.



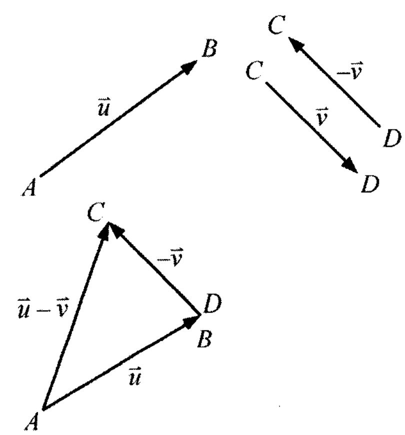
The Negative of a Vector

The negative of a vector, $\vec{v}$, is denoted by $-\vec{v}$. The vector $-\vec{v}$ is equal in magnitude to $\vec{v}$ but in the opposite direction. The negative of $\overrightarrow{AB}$ is $-\overrightarrow{AB}$, which is equal to $\overrightarrow{BA}$.

Subtracting Geometric Vectors

To subtract a geometric vector in two space, add its negative vector.

In the given diagram, $\overrightarrow{AB} + (-\overrightarrow{CD}) = \overrightarrow{AB} + \overrightarrow{DC} = \overrightarrow{AC}$.



Multiplying Geometric Vectors by a Scalar

When a vector, $\vec{v}$, is multiplied by a scalar, k, then the magnitude, or length, of the vector $k\vec{v}$ is $|k|$ times the magnitude of $\vec{v}$.

$|k\vec{v}| = |k||\vec{v}|$

The direction of $k\vec{v}$ is the same as $\vec{v}$ when $k > 0$ and opposite when $k < 0$.

Adding and Subtracting Vectors in Cartesian Form

Add or subtract algebraic vectors in two and three space in Cartesian form by adding or subtracting their x-, y-, and z-components.

- If $\vec{u} = (x_1, y_1)$ and $\vec{v} = (x_2, y_2)$, then $\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2)$ and $\vec{u} - \vec{v} = (x_1 - x_2, y_1 - y_2)$.
- If $\vec{u} = (x_1, y_1, z_1)$ and $\vec{v} = (x_2, y_2, z_2)$, then $\vec{u} + \vec{v} = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$ and $\vec{u} - \vec{v} = (x_1 - x_2, y_1 - y_2, z_1 - z_2)$.

Multiplying Algebraic Vectors by a Scalar

When multiplying algebraic vectors by a scalar, k, if $\vec{u} = (x_1, y_1)$, then $k\vec{u} = (kx_1, ky_1)$. If $\vec{u} = (x_1, y_1, z_1)$, then $k\vec{u} = (kx_1, ky_1, kz_1)$.

8. If $\vec{u} = (2, -1, \sqrt{2})$, how can $-\sqrt{8}\vec{u}$ be represented?

A. $(-4\sqrt{2}, -2\sqrt{2}, -4)$

B. $(-4\sqrt{2}, 2\sqrt{2}, -4)$

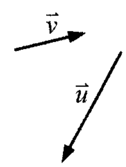
C. $(-4, -2\sqrt{2}, -4)$

D. $(-4, 2\sqrt{2}, -4)$

Numerical Response

9. Given $\vec{p} = (-h, 2k)$ and $\vec{q} = (-4h, -3k)$, if $2\vec{p} - 3\vec{q}$ is expressed as (ah, bk), then the sum of a and b is _________.

Use the following information to answer the next question.



Two vectors are given.

Written Response

10. Given the two vectors shown, sketch the vector $\vec{u} - 2\vec{v}$.

VEC2.2 determine, through investigation with and without technology, some properties (e.g., commutative, associative, and distributive properties) of the operations of addition, subtraction, and scalar multiplication of vectors

Investigating Properties of Operations on Vectors

The following properties for vector addition and scalar multiplication exist for any of the vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in two or three space and the scalars $m, n \in \mathbb{R}$:

- The commutative property, $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- The associative property, $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$, $(mn)\vec{v} = m(n\vec{v})$
- The distributive property, $m(\vec{u} + \vec{v}) = m\vec{u} + m\vec{v}$
- The identity property, $\vec{v} + \vec{0} = \vec{v}$, $1\vec{v} = \vec{v}$

Written Response

11. Given the vectors $\vec{u} = (1, 1, 1)$, $\vec{v} = (-3, -2, 4)$, and $\vec{w} = (2, 3, -1)$, determine algebraically if the associative property for vector subtraction, $(\vec{u} - \vec{v}) - \vec{w} = \vec{u} - (\vec{v} - \vec{w})$, is true.

VEC2.3 solve problems involving the addition, subtraction, and scalar multiplication of vectors, including problems arising from real-world applications

Applications With Vectors

There are many applications of vectors in mathematics, science, and engineering. The most fundamental applications have physical quantities such as force and velocity, which have both magnitude and direction.

These terms are often used in application problems:

1. **Resultant**—a single force that has the same effect as the given forces. The resultant is often symbolized by $\vec{R}$.
2. **Equilibrant**—a single force that maintains a body in a state of equilibrium. The equilibrant is often symbolized by $\vec{E}$. The equilibrant is equal in magnitude but opposite in direction to the resultant.

Example

An airplane is heading on a bearing of 60° at 200 km/h. A wind of 75 km/h is blowing from a bearing of 340°.

To the nearest whole number, determine the airplane's velocity and its direction of travel.

Solution

Step 1

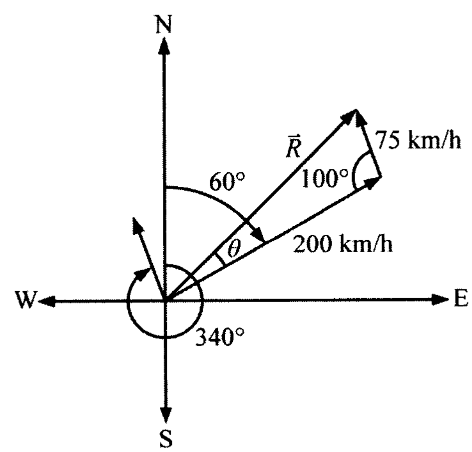
Sketch a diagram with representative vectors on a coordinate system.

The vectors do not need to be drawn exactly to scale, nor do the angular directions need to be measured. The purpose of the diagram is to represent the situation and determine the relative positions of the given vectors and the resultant.

A vector equal to the wind vector is drawn with its tail at the head of the airplane vector.

A resultant vector $\vec{R}$ is then drawn from the tail of the airplane vector to the head of the new wind vector. The vectors are labelled with their magnitudes.

If the angle between the tails of the two vectors is 80°, then the angle between the head and tail of the two vectors is 180° – 80° = 100°.



Step 2

Determine the velocity of the airplane.

The velocity is represented by the resultant vector.

Apply the cosine law to find the magnitude of the resultant.

$$a^2 = b^2 + c^2 - 2ab\cos A$$
$$|\vec{R}|^2 = 200^2 + 75^2 - (200)(75)\cos 100°$$
$$|\vec{R}|^2 \approx 40\,000 + 5\,625 - 15\,000(-0.1736)$$
$$|\vec{R}|^2 \approx 48\,229.7$$
$$|\vec{R}| \approx \sqrt{48\,229.7} \approx 219.6$$

Step 3

Determine the direction of the airplane.

The angle θ is determined by applying the sine law.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin B}$$
$$\frac{75}{\sin\theta} \approx \frac{219.6}{\sin 100°}$$
$$\sin\theta \approx \frac{75\sin 100°}{219.6} \approx \frac{75(0.9848)}{219.6} \approx 0.3363$$
$$\theta \approx 19.7$$

The velocity of the airplane is approximately 220 km/h, and the direction of travel is approximately on a bearing of 60° – 20° = 40°.

Use the following information to answer the next multipart question.

12. An airplane is heading on a bearing of 125° at 400 km/h. A wind is blowing from a bearing of 225° at 200 km/h.

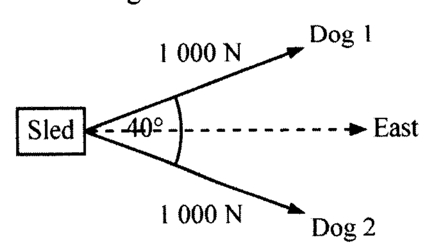
Written Response

a) Determine the magnitude of the resultant ground velocity ($\vec{R}$) of the airplane to the nearest kilometre per hour.

b) Determine the direction (as a bearing) of the resultant ground velocity of the airplane to the nearest degree.

Use the following information to answer the next multipart question.

13. Two dogs pull a heavy sled due east. Each dog pulls with a force of 1 000 N, where the angle between them is 40.



a) To the nearest newton, the magnitude of the resultant force on the sled is

A. 1 414 N **B.** 1 532 N

C. 1 879 N **D.** 2 000 N

b) What is the magnitude of the southerly force with which a third dog would have to pull so that the resultant force due to all three dogs would be 40°?

A. 1 186 N **B.** 1 208 N

C. 1 577 N **D.** 1 678 N

VEC2.4 perform the operation of dot product on two vectors represented as directed line segments (i.e., using vector a – vector b = vector ab cosine θ) and in Cartesian form (i.e., using vector a – vector b = a^1b^1 plus a^2b^2 or vector a minus vector b equals a^1b^1 plus a^2b^2 plus a^3b^3) in two-space and three-space, and describe applications of the dot product (e.g., determining the angle between two vectors; determining the projection of one vector onto another)

VEC2.5 determine, through investigation, properties of the dot product (e.g., investigate whether it is commutative, distributive, or associative; investigate the dot product of a vector with itself and the dot product of orthogonal vectors)

VEC2.8 solve problems involving dot product and cross product (e.g., determining projections, the area of a parallelogram, the volume of a parallelepiped), including problems arising from real-world applications (e.g., determining work, torque, ground speed, velocity, force)

Properties and Applications of the Dot Product

The dot product is a scalar and not a vector. It is sometimes called the **scalar product**.

- If $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$, the dot product $\vec{a} \cdot \vec{b}$ is defined as $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2$.
- If $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$, the dot product $\vec{a} \cdot \vec{b}$ is defined as $\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.

The properties of the dot product apply to all algebraic vectors in two space and three space. The dot product has the following properties:

- The dot product is commutative: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$.
- The dot product is distributive over addition: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$.
- The dot product is not associative: $(\vec{a} \cdot \vec{b}) \cdot \vec{c} \neq \vec{a} \cdot (\vec{b} \cdot \vec{c})$.
- The dot product of a vector with itself is equal to the square of its magnitude: $\vec{a} \cdot \vec{a} = |\vec{a}|^2$.
- Two non-zero vectors $\vec{a}$ and $\vec{b}$ are orthogonal if and only if $\vec{a} \cdot \vec{b} = 0$.

The term *orthogonal* is used to refer to two non-zero vectors that have an angle of 90°.

The dot product in two and three space has several applications. These include determining the angles between two vectors, determining the projection of one vector onto another, and calculating work.

Determining the Angle between Two Vectors

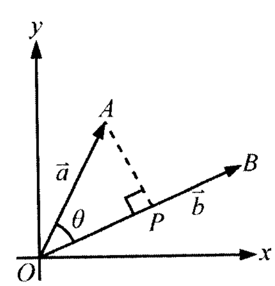
The angle between two vectors can be found using the dot product. For any two vectors with angle θ between them, $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$, or

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \cos\theta.$$

This relationship holds for vectors in two space or three space.

Determining the Projection of One Vector onto Another

The dot product can be used to determine the projection of one vector onto another. The **vector projection** of $\vec{a}$ onto $\vec{b}$ is the projection of the line segment representing $\vec{a}$ on the line segment representing $\vec{b}$. In the given diagram, the vector projection of $\vec{a}$ onto $\vec{b}$ is the vector component of $\vec{a}$, namely $\overrightarrow{OP}$, in the direction of $\vec{b}$.



Therefore, $\overrightarrow{OP} = \text{proj}_{\vec{b}}\vec{a}$.

The vector projection of $\vec{a}$ onto $\vec{b}$ can be expressed as $\text{proj}_{\vec{b}}\vec{a} = \left(\frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}}\right)\vec{b}$.

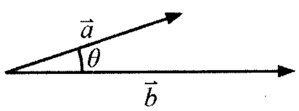
The magnitude of the projection is also called the **scalar projection** of $\vec{a}$ onto $\vec{b}$. It can be expressed as $|\text{proj}_{\vec{b}}\vec{a}| = |\vec{a}| \cos\theta$ and

$$|\text{proj}_{\vec{b}}\vec{a}| = \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|}.$$

Calculating Work

Work is a force that is applied over a particular distance. Work is a scalar quantity. When the force is in newtons and the distance is in metres, the work done is in joules.

Consider a force $\vec{a}$ acting on an object being displaced in the direction of $\vec{b}$ at an angle θ.



The work done on the object is the product of the magnitude of the force acting in the direction of motion and the magnitude of the displacement travelled by the object.

$$\begin{aligned} \text{work} &= \left(\begin{array}{c}(\text{magnitude of force along } |\vec{b}|) \\ (\text{magnitude of displacement})\end{array}\right) \\ &= |\text{proj}_{\vec{b}}\vec{a}| \times |\vec{b}| \\ &= \frac{|\vec{a} \cdot \vec{b}|}{|\vec{b}|} \times |\vec{b}| \\ &= |\vec{a} \cdot \vec{b}| \end{aligned}$$

Use the following information to answer the next question.

Vectors $\vec{u} = (-3, b, 2)$ and $\vec{v} = (5, 2, 7)$ are given. The dot product of these vectors $(\vec{u} \cdot \vec{v})$ is 15.

Numerical Response

14. The value of b is ________.

Written Response

15. Determine the values of a and b so that the vector $\vec{u} = (1, 2, 3)$ is orthogonal to both vectors $\vec{v} = (a, b, 5)$ and $\vec{w} = (3, a, b)$.

Use the following information to answer the next multipart question.

16. Two forces, $\vec{F}_1 = (200, 500, 450)$ and $\vec{F}_2 = (200, -400, 300)$, both measured in newtons, act on an object through a displacement defined by $\vec{d} = (4, 2, 13)$, measured in metres.

Written Response

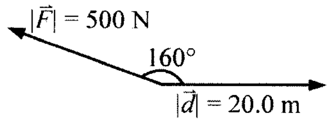
a) Use the concept of vector projection to determine the total magnitude of the force applied on the object in the direction of its displacement, to the nearest newton.

b) Using the dot product, find the total work done on the object to the nearest joule.

c) Given that gravity is a force in the direction of the negative z-axis, determine the total work done against gravity to the nearest joule.

Use the following information to answer the next question.

A frictional braking force is applied at an angle to an object. The frictional braking force is 500 N, and the angle it is applied at is 160°. The object is moving forward through a displacement of 20.0 m.



17. To the nearest joule, the work done by the braking force is

A. −10 000 J **B.** −9 397 J

C. 9 397 J **D.** 10 000 J

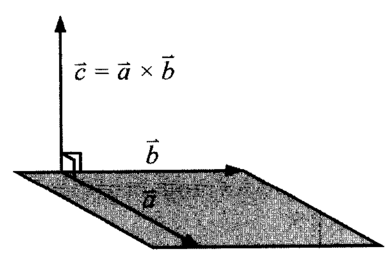
VEC2.6 perform the operation of cross product on two vectors represented in Cartesian form in three-space [i.e., using vector a x vector b = ($a^2b^3 - a^3b^2, a^3b^1 - a^1b^3, a^1b^2 - a^2b^1$)], determine the magnitude of the cross product (i.e., using | vector a x vector b | = | vector a | | vector b | sin θ), and describe applications of the cross product (e.g., determining a vector orthogonal to two given vectors; determining the turning effect [or torque] when a force is applied to a wrench at different angles)

VEC2.7 determine, through investigation, properties of the cross product (e.g., investigate whether it is commutative, distributive, or associative; investigate the cross product of collinear vectors)

VEC2.8 solve problems involving dot product and cross product (e.g., determining projections, the area of a parallelogram, the volume of a parallelepiped), including problems arising from real-world applications (e.g., determining work, torque, ground speed, velocity, force)

Properties and Applications of the Cross Product

The cross product of two vectors in three space produces a new vector that is perpendicular (orthogonal) to the two original vectors. In this diagram, the cross product $\vec{a} \times \vec{b}$ is shown as vector $\vec{c}$.



If $\vec{a}$ has components (a_1, a_2, a_3) and $\vec{b}$ has components (b_1, b_2, b_3), then the cross product $\vec{a} \times \vec{b}$ has components $(a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1)$.

The magnitude of the cross product $\vec{a} \times \vec{b}$ is given by the formula $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$, where θ is the angle between vectors $\vec{a}$ and $\vec{b}$.

Properties of the Cross Product

The cross product has the following properties:

- The cross product is not commutative: $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$.
- The cross product is distributive over addition: $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$.
- The cross product is not associative: $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$.
- The cross product of collinear vectors is the zero vector (0, 0, 0).

Torque and the Cross Product

A basic application of the cross product is a physics situation that involves the turning effect or torque. **Torque** is a vector quantity, $\vec{\tau}$, that is equal to the cross product of the lever arm vector, $\vec{l}$, and the applied force vector, $\vec{F}$.

$\vec{\tau} = \vec{l} \times \vec{F}$

The magnitude of the torque vector can be calculated as follows:

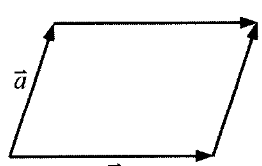
$|\vec{\tau}| = |\vec{l} \times \vec{F}|$

$|\vec{\tau}| = |\vec{l}|\,|\vec{F}|\sin\theta$

Geometric Applications of the Cross Product

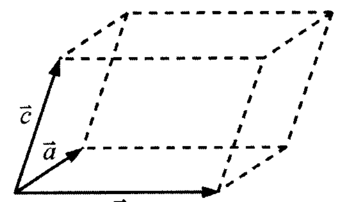
Another application of the dot and cross products relates to geometric problems.

The area of a parallelogram whose sides are represented by the vectors $\vec{a}$ and $\vec{b}$ can be found by determining the magnitude of the cross product so that $A = |\vec{a} \times \vec{b}|$.



$A = |\vec{a} \times \vec{b}|$

A parallelepiped is a prism whose base is a parallelogram. If the sides are represented by the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$, the volume is given by the formula $V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$.



$V = |\vec{a} \cdot (\vec{b} \times \vec{c})|$

CHALLENGER QUESTION

18. What is the measure of the angle θ between the vectors $\vec{u} = (6, 2, 1)$ and $\vec{v} = (-4, 1, 2)$?

A. 43° **B.** 47°

C. 133° **D.** 137°

19. For the cross product between vectors $\vec{a} = (h, 3, -1)$ and $\vec{b} = (k, 1, -2)$ to be $\vec{a} \times \vec{b} = (-5, 7, 1)$, the values of h and k should be

A. $h = 1$ and $k = 4$

B. $h = 4$ and $k = 1$

C. $h = \frac{9}{5}$ and $k = \frac{-22}{5}$

D. $h = \frac{-22}{5}$ and $k = \frac{9}{5}$

Use the following information to answer the next question.

Josh wanted to show that $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$. He wrote three sets of vectors: $\vec{a}$, $\vec{b}$, and $\vec{c}$.

I. $\vec{a} = (1, 0, 0)$, $\vec{b} = (0, 1, 0)$, $\vec{c} = (0, 0, 1)$
II. $\vec{a} = (1, 1, 0)$, $\vec{b} = (1, -1, 0)$, $\vec{c} = (0, 0, -1)$
III. $\vec{a} = (1, 0, 0)$, $\vec{b} = (1, 1, 0)$, $\vec{c} = (1, 1, 1)$

20. Which of the given sets of vectors supports Josh's statement that $(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$?
A. Sets II and III only
B. Sets I and II only
C. Sets I, II, and III
D. Set III only

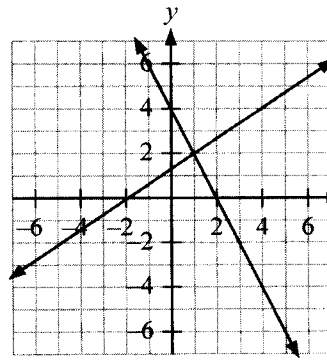
21. The volume of a parallelepiped defined by the vectors $\vec{a} = (2, -5, -1)$, $\vec{b} = (4, 0, 1)$, and $\vec{c} = (3, -2, -2)$ is
A. 26 units^3 **B.** 28 units^3
C. 37 units^3 **D.** 43 units^3

VEC3.1 recognize that the solution points (x, y) in two-space of a single linear equation in two variables form a line and that the solution points (x, y) in two-space of a system of two linear equations in two variables determine the point of intersection of two lines, if the lines are not coincident or parallel

Systems of Two Linear Equations in Two Space

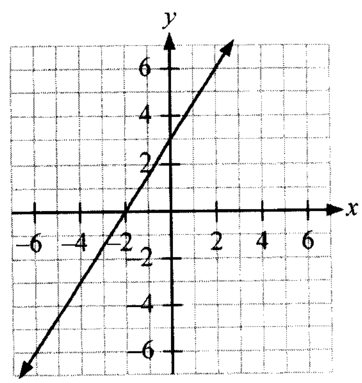
A system of two equations in two unknowns (x and y) produces one of the following results:

1. The solution to the system consists of one unique ordered pair of the variables (x_1, y_1). When this occurs, the graph of the system consists of a pair of lines intersecting at the point (x_1, y_1), as shown in the given example.



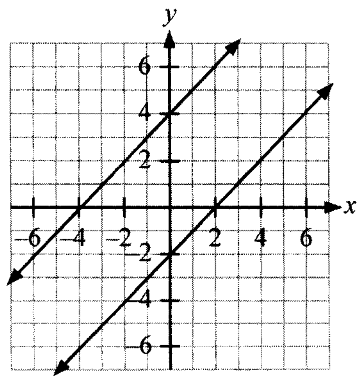
The equations of the lines will have different slope values (m) when they are written in the form $y = mx + b$.

2. The solution to the system consists of all ordered pairs (x, y) that satisfy either equation. In this case, the graph of the system consists of coincident lines (one line), as shown here.



The equations of the lines can be written as identical equations

3. There is no solution to the system; that is, the solution is the empty set. When there is no solution to the system, the graph of the system consists of parallel lines.



The equations of the lines will have the same slope value (m) but different y-intercept values (b) when written in the form $y = mx + b$.

Use the following information to answer the next question.

l_1: $2x + 3y - 18 = 0$
l_2: $5x + \frac{15}{2}y - 210 = 0$

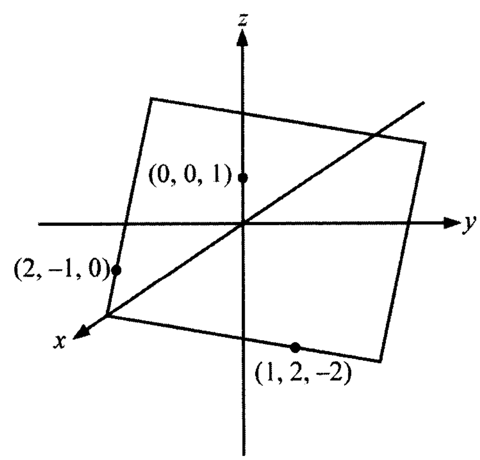
Written Response

22. Use algebra to explain whether the two given lines are coincident, parallel, or intersecting.

VEC3.2 determine, through investigation with technology (i.e., 3-D graphing software) and without technology, that the solution points (x, y, z) in three-space of a single linear equation in three variables form a plane and that the solution points (x, y, z) in three-space of a system of two linear equations in three variables form the line of intersection of two planes, if the planes are not coincident or parallel

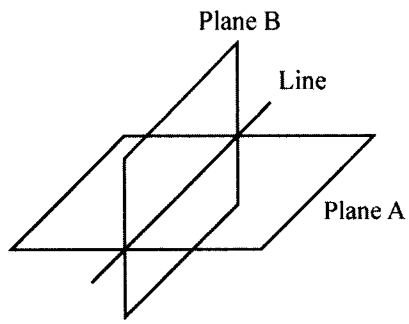
Investigating Solution Points of Linear Equations in Three Space

In the same way that two points determine a line in two space, three points that are not collinear determine a plane in three space. If the points are the endpoints of geometric vectors in three space represented by algebraic vectors of the form (x, y, z), then an equation in three variables describes all points in the plane. For example, the points (1, 0, 0), (0, 1, 0), and (0, 0, 1) satisfy the equation $x + y + z = 1$. All points that satisfy $x + y + z = 1$ lie in the same plane, such as points (2, –1, 0) and (1, 2, –2).

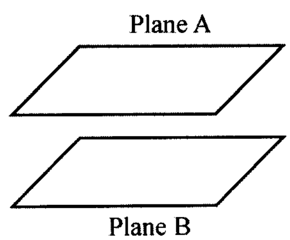


Two linear equations that each have three variables will create a graph consisting of two planes. The planes are oriented in one of the following ways:

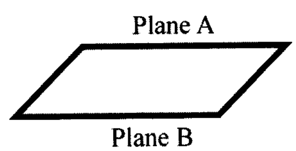
1. The planes intersect to form a line.



2. The planes are parallel.



3. The planes are coincident.



23. What is the geometric relationship between the two planes defined by the equations $4x - y + z - 3 = 0$ and $3x + y + z - 3 = 0$?

A. They are parallel.

B. They are coincident.

C. They intersect in a line.

D. They intersect at a point.

Use the following information to answer the next question.

The points (2, 0, 0), (0, 4, 1), and (1, 4, 5) lie on the plane defined by $Ax + By + Cz + D = 0$, where $A \neq 0$.

24. What are the values of B, C, and D in terms of A?

A. $B = -\frac{9}{16}A, C = -4A, D = -2A$

B. $B = \frac{9}{16}A, C = -\frac{1}{4}A, D = -2A$

C. $B = -\frac{9}{16}A, C = \frac{1}{4}A, D = -2A$

D. $B = \frac{9}{16}A, C = \frac{1}{4}A, D = 2A$

VEC3.3 determine, through investigation using a variety of tools and strategies (e.g., modelling with cardboard sheets and drinking straws; sketching on isometric graph paper), different geometric configurations of combinations of up to three lines and/or planes in three-space (e.g., two skew lines, three parallel planes, two intersecting planes, an intersecting line and plane); organize the configurations based on whether they intersect and, if so, how they intersect (i.e., in a point, in a line, in a plane)

Geometric Configurations of Lines and Planes in Three Space

There are many possible geometric configurations of up to three lines and/or planes in three space. Configurations that have common intersection points between all the lines and/or planes represent systems of equations that have solutions. The types of solutions that are possible can be described as follows:

- A system has an infinite number of solutions and is said to be **consistent** and **dependent** when the lines and/or planes intersect in a line or plane.
- A system has a unique solution and is said to be **consistent** and **independent** when the lines and/or planes intersect in one point.
- A system has no solution points and is said to be **inconsistent** when the lines and/or planes do not intersect.

The following tables summarize the possible configurations and the solutions to the corresponding systems of equations for two lines and three planes in three space.

Two Lines in Three Space

Configuration	Solution (Type of System)
Intersect in one point	One (consistent and independent)
Coincident lines	Infinite (consistent and dependent)
Parallel lines	None (inconsistent)
Skew lines	None (inconsistent)

Three Planes in Three Space

Configuration	Solution (Type of System)
Intersect in one point	One (consistent and independent)
Intersect in a line	Infinite (consistent and dependent)
Three coincident planes	Infinite (consistent and dependent)
Two coincident planes intersecting a third in a line	Infinite (consistent and dependent)
One plane intersecting two parallel planes in two parallel lines	None (inconsistent)
Three distinct planes intersecting in three parallel lines	None (inconsistent)
Three distinct parallel planes	None (inconsistent)
Two coincident planes parallel to a third plane	None (inconsistent)

25. Which of the following statements about the possible intersections of three different non-parallel planes in three space is **false**?

A. The planes could intersect at a single point.

B. The planes could intersect in a single line.

C. The planes could intersect in three lines.

D. The planes could intersect at two points.

VEC4.1 recognize a scalar equation for a line in two-space to be an equation of the form Ax + By + C = 0, represent a line in two-space using a vector equation (i.e., vector r = vector r 0 + t vector m.) and parametric equations, and make connections between a scalar equation, a vector equation, and parametric equations of a line in two-space

Scalar, Vector, and Parametric Equations of Lines in Two Space

The equation of a line in two space can be written in one of the following three forms:

1. Scalar: $Ax + By + C = 0$
2. Vector: $\vec{r} = \vec{r}_0 + t\vec{m}$
3. Parametric: $x = x_0 + at$, $y = y_0 + bt$

The scalar equation of a line can be determined from the parametric form by solving both equations for t, then equate the two equations and rearrange them into the form $Ax + By + C = 0$.

The vector equation of a line can be determined from the scalar form by finding any point on the line, $Ax + By + C = 0$, that can be used as the position vector, $\vec{r}_0$. Next, determine the slope of the line, $Ax + By + C = 0$, by rewriting the equation in slope-intercept form. The slope gives the direction vector, $\vec{m} = (x, y)$. Substitute the position and direction vector into the form $\vec{r} = \vec{r}_0 + t\vec{m}$.

The parametric equations of a line can be determined from the vector form where $(x, y) = (x_0, y_0) + t(a, b)$. Therefore,

$x = x_0 + at$
$y = y_0 + bt$

Use the following information to answer the next question.

Four students each made a statement about the equation of a line in two space given in vector form, $\vec{r} = (-3, 5) + t(2, -3)$.

- Jeremy: The equation of this line in scalar form is $3x + 2y - 1 = 0$.
- Derek: A line, L, that is parallel to the given line is as follows:
  $$L : \begin{cases} x = 1 + 2.5t \\ y = 8 - 3.75t \end{cases}$$
- Charissa: A vector that is perpendicular to the given line is $\vec{v} = (9, -4)$.
- Emily: Two points on this line are $(-1, 2)$ and $(0, 0.5)$.

26. Which of the four given statements is **false**?

A. Emily's **B.** Derek's

C. Jeremy's **D.** Charissa's

VEC4.2 recognize that a line in three-space cannot be represented by a scalar equation, and represent a line in three-space using the scalar equations of two intersecting planes and using vector and parametric equations (e.g., given a direction vector and a point on the line, or given two points on the line)

Representing Lines in Three Space

A line in three space cannot be represented by a scalar equation. A line in three space is formed from the intersection of two non-parallel or non-coincident planes, so a line in three space represents the solution to a system of two scalar equations with three unknowns.

Although it is not possible to represent a line in three space with a single scalar equation, it is possible to represent a line using a vector equation or a set of three parametric equations. The vector equation of the form $\vec{r} = \vec{r}_0 + t\vec{m}$ can be applied, where $\vec{r}_0 = (x_0, y_0, z_0)$ represents a position vector, $\vec{m} = (a, b, c)$ represents a direction vector, and both vectors are in three space. The set of three parametric equations is of the form

$x = x_0 + at$
$y = y_0 + bt$
$z = z_0 + ct$

Use the following information to answer the next question.

A line in three space is defined by the vector equation $\vec{r} = (2, 1, 0) + t(-3, 0, 1)$.

27. Which of the following pairs of intersecting planes would also define this line?

A. P_1: $y - 1 = 0$
P_2: $x - y - 1 = 0$

B. P_1: $y - 1 = 0$
P_2: $x + 3z - 2 = 0$

C. P_1: $y - 1 = 0$
P_2: $3x + y - 7 = 0$

D. P_1: $y - 1 = 0$
P_2: $-2x + 4y + z = 0$

28. Which of the following points does **not** lie on the line that passes through the points $P(6, -2, 4)$ and $Q(-1, -4, 0)$?

A. $(-8, -6, -4)$

B. $\left(\frac{23}{2}, -7, -6\right)$

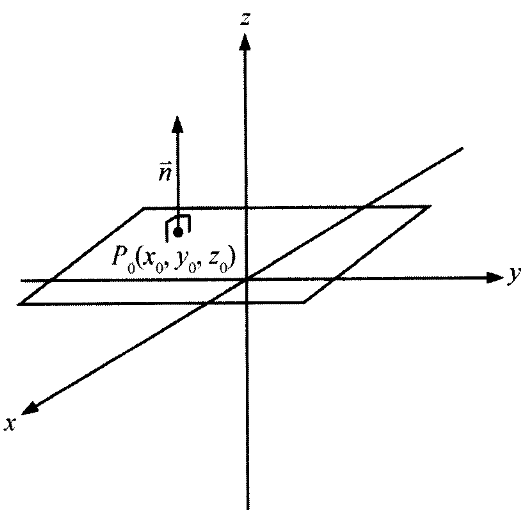
C. $\left(\frac{33}{2}, 1, 10\right)$

D. $(13, 0, 8)$

VEC4.3 recognize a normal to a plane geometrically (i.e., as a vector perpendicular to the plane) and algebraically, and determine, through investigation, some geometric properties of the plane (e.g., the direction of any normal to a plane is constant; all scalar multiples of a normal to a plane are also normals to that plane; three non-collinear points determine a plane; the resultant, or sum, of any two vectors in a plane also lies in the plane)

PROPERTIES OF THE NORMAL OF A PLANE

From the given configuration, you can see how a particular plane is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\vec{n}$ orthogonal (perpendicular) to the plane at that point.



A vector that is orthogonal to a plane is called a **normal vector**.

The following are some properties that relate planes and normal vectors.

- A plane with the equation $Ax + By + Cz + D = 0$ has a normal vector (A, B, C).
- The direction of any normal to a plane is constant.
- All scalar multiples of a normal to a plane are also normal to a plane.
- Any three non-collinear points determine a plane.
- The resultant (sum) of two vectors that lie in a plane also lies in the same plane.

29. Which of the following vectors is **not** normal to the plane defined by the scalar equation $3x - 4y + z - 12 = 0$?

A. $\vec{u} = \left(-\frac{9}{4}, 3, -\frac{3}{4}\right)$

B. $\vec{v} = (-3, 4, -1)$

C. $\vec{w} = \left(-1, \frac{4}{3}, \frac{1}{3}\right)$

D. $\vec{t} = (3, -4, 1)$

Use the following information to answer the next multipart question.

30. The scalar equation of a plane is given as $-2x + 3y - 4z + 12 = 0$.

Written Response

a) Is vector $\vec{u} = (5, 6, 2)$ parallel to the plane? Explain your answer.

b) Show that the resultant of any two vectors that lie on the plane also lies on the plane.

VEC4.4 recognize a scalar equation for a plane in three-space to be an equation of the form Ax + By + Cz + D = 0 whose solution points make up the plane, determine the intersection of three planes represented using scalar equations by solving a system of three linear equations in three unknowns algebraically (e.g., by using elimination or substitution), and make connections between the algebraic solution and the geometric configuration of the three planes

Determining the Intersection of Three Planes

A plane can be represented by a scalar equation in three space of the form $Ax + By + Cz + D = 0$. The solutions points, (x, y, z), of the equation make up the plane.

The following table summarizes the relationships between the configuration of three planes in three space, the solution to the system, and the orientation of the normal vectors.

Three Planes in Three Space

Configuration	Solution	Normal Vectors	
		Parallel	**Coplanar**
Intersect in one point	Point	No	No
Intersect in a line	Line	No	Yes
Three coincident planes	Plane	Yes	Yes
Two coincident planes intersecting a third in a line	Line	Two only	Yes
One plane intersecting two parallel planes in two parallel lines	None	Two only	Yes
Three distinct planes intersecting in three parallel lines	None	No	Yes
Three distinct parallel planes	None	Yes	Yes
Two coincident planes parallel to a third plane	None	Yes	Yes

Determining normal vectors to the planes represented by the equations can assist in determining the type of system of equations that is involved. Two properties of the normal vectors are particularly useful:

1. Planes are parallel if and only if their normal vectors are scalar multiples of each other.
2. Normal vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ are coplanar if and only if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$. If $\vec{a} \cdot (\vec{b} \times \vec{c}) \neq 0$, the planes intersect in a single point.

When solving a system of three equations in three unknowns, the normal vectors can be used to determine whether the system appears to have a solution; however, other than checking for scalar multiples, time spent working with the normal vectors can often be better spent solving the system. The algebraic method of elimination or substitution can be used to determine the solution values.

To solve a linear system using the elimination method, follow these steps:

1. Label the equations, and choose a variable to be eliminated.
2. Eliminate the same variable from any two pairs of the three given equations by adding or subtracting the equations in the system
3. Eliminate another variable by adding or subtracting the two new equations.
4. Solve for the remaining variable.
5. Substitute the solved value into an equation from the step 3, and then solve for the other variable.
6. Substitute the values back into one of the original equations, and solve for the last variable.
7. State the solution as an ordered triplet, (x, y, z).

To solve a system of linear equations using the substitution method, follow these steps:

1. Write one of the equations so that a variable is isolated in terms of an expression of the other variables.
2. Substitute this expression into the other two equations so they can be rewritten as a first-degree equation in two variables.
3. Write one of the remaining two equations so that a variable is isolated in terms of an expression of the other variable.
4. Substitute this expression into the other equation so it can be solved as a first-degree equation in one variable.
5. Substitute the solved value in the equation from the previous step, and then solve for the other variable.
6. Substitute the solved values into any one of the original equations, and then solve for the last variables.
7. Write the solution as an ordered triplet, (x, y, z).

Use the following information to answer the next question.

Three planes are given.
P_1: $6x - 4y + 2z - 2 = 0$
P_2: $3x + 2y - z - 6 = 0$
P_3: $9x - 6y + 3z + 8 = 0$

31. Which of the following statements about the configuration of these planes is **true**?

A. The three planes are parallel and distinct.

B. Two planes are parallel, and the other is distinct.

C. The planes are non-parallel and intersect at a line.

D. The planes are non-parallel and intersect at a point.

Use the following information to answer the next question.

The following system of three planes is consistent:
$x + 4y + 3z - 5 = 0$
$x + 3y + 2z - 4 = 0$
$x + y - z + 1 = 0$

Written Response

32. Solve the given system of equations algebraically (i.e., elimination, substitution), and describe its solution.

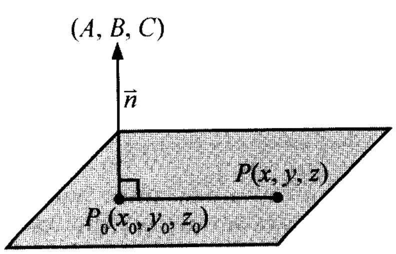
VEC4.5 determine, using properties of a plane, the scalar, vector, and parametric equations of a plane

Equations Defining a Plane

Just as there are different forms of equations defining lines in two space, there are various forms of equations that define planes in three space.

The Scalar Equation of a Plane

Recall that a particular plane can be determined by a point on the plane and a normal vector to the plane.



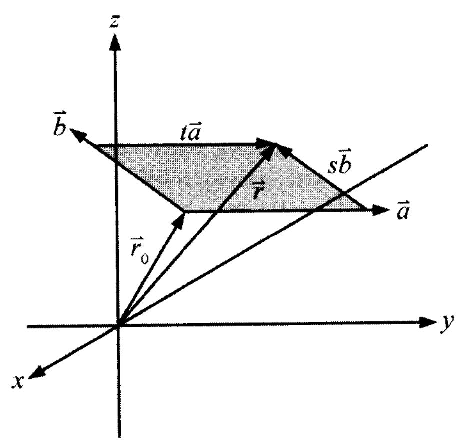
If a particular point $P_0(x_0, y_0, z_0)$ and any second point $P(x, y, z)$ are on the plane, then vector $\overrightarrow{P_0P}$ is $(x - x_0, y - y_0, z - z_0)$. If the normal vector $\vec{n}$ has coordinates (A, B, C), then because $\vec{n}$ is perpendicular to $\overrightarrow{P_0P}$, $(A, B, C) \cdot \overrightarrow{P_0P} = 0$. Thus, substituting $(x - x_0, y - y_0, z - z_0)$ for $\overrightarrow{P_0P}$ in this equation gives the following result:

$$(A, B, C) \cdot (x - x_0, y - y_0, z - z_0) = 0$$
$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$
$$Ax + By + Cz + (-Ax_0 - By_0 - Cz_0) = 0$$

Notice that $-Ax_0 - By_0 - Cz_0$ is a constant. Using D to represent this constant gives $Ax + By + Cz + D = 0$ as the equation of a plane with the normal vector $\vec{n} = (A, B, C)$. The value of D is determined by substituting a known point on the plane into the equation $Ax + By + Cz + D = 0$ and solving for D.

The Vector Equation of a Plane

It is possible to describe all points on a plane, and thus the plane itself, as algebraic vectors that are the linear combination of an algebraic vector on the plane and two directional vectors in the plane.



In the diagram shown, $\vec{r}$ is a linear combination of $\vec{r}_0$, $\vec{a}$, and $\vec{b}$ such that $\vec{r} = \vec{r}_0 + t\vec{a} + s\vec{b}$, where t and s are real number scalars.

Since vectors $\vec{r}_0$, $\vec{a}$, and $\vec{b}$ can be represented by ordered triplets, the vector form of the equation of a plane is $(x, y, z) = (x_0, y_0, z_0) + t(a_1, a_2, a_3) + s(b_1, b_2, b_3)$.

The Parametric Equations of a Plane

The parametric equations of a plane can be determined from the vector form $(x, y, z) = (x_0, y_0, z_0) + t(a_1, a_2, a_3) + s(b_1, b_2, b_3)$.

$x = x_0 + ta_1 + sb_1$
$y = y_0 + ta_2 + sb_2$
$z = z_0 + ta_3 + sb_3$

These equations represent the parametric form of the equation of a plane. Note that since different choices of $\vec{r}_0$, $\vec{a}$, and $\vec{b}$ can be used to determine the same plane, the vector and parametric forms are not unique.

33. A plane passes through the points $A(1, 12, 5)$, $B(-2, -6, 4)$, and $C(-1, -1, -4)$. A vector equation representing this plane is

A. $(x, y, z) = \begin{pmatrix} (1, 12, 5) \\ + t(3, 18, 1) \\ + s(2, 13, -9) \end{pmatrix}$

B. $(x, y, z) = \begin{pmatrix} (-2, -6, 4) \\ + t(-3, -18, 1) \\ + s(1, 5, 8) \end{pmatrix}$

C. $(x, y, z) = \begin{pmatrix} (-2, -6, 4) \\ + t(-3, -18, 1) \\ + s(-1, 5, -8) \end{pmatrix}$

D. $(x, y, z) = \begin{pmatrix} (1, 12, 5) \\ + t(-3, -18, -1) \\ + s(-2, -13, -9) \end{pmatrix}$

34. A plane contains the points $P(4, 6, -2)$, $Q(-2, 0, -8)$, and $R(-10, -2, 2)$. Which of the following statements about this plane is **true**?

A. A parametric equation of the plane is $x = 4 + 6t + 8s$, $y = 6 + 6t - 2s$, and $z = -2 + 6t - 10s$.

B. The scalar equation of the plane is $2x - 3y + z - 12 = 0$.

C. The x-intercept of the plane is $(-6, 0, 0)$.

D. A point on the plane is $(-2, 1, -7)$.

VEC4.6 determine the equation of a plane in its scalar, vector, or parametric form, given another of these forms

Converting Forms of Equations of Planes

Equations representing planes can be converted to different forms by applying various properties and concepts related to planes.

Converting from Scalar Form to Vector or Parametric Form

In order to write the equation of a plane in parametric or vector form, it is necessary to determine a point on the plane and a pair of non-parallel direction vectors that are in the plane. There are a couple of choices for obtaining the point and required vectors. Convenient points on the plane can be found by having two of the three coordinates equal to zero. Any two out of three non-collinear points can be used to obtain direction vectors. As well, a plane of the form $Ax + By + Cz + D = 0$ has a normal vector $\vec{n} = (A, B, C)$. The cross product of $\vec{n} = (A, B, C)$ and any vector in the plane will yield another vector in the plane that will be perpendicular to both of them.

Converting from Vector or Parametric Form to Scalar Form

One possible approach that can be used to convert an equation from vector or parametric form to scalar form is to reverse the order of the procedure for changing an equation from scalar form to vector or parametric form.

From the vector or parametric form, the coordinates of a point and two direction vectors can be obtained. The cross product of the direction vectors creates a vector that is normal to the plane. The coordinates of the normal vector are the values of A, B, and C in $Ax + By + Cz + D = 0$. The value of D is obtained by substituting the coordinates of the known point for x, y, and z.

Use the following information to answer the next question.

The scalar form of the equation of a plane is $Ax - 10y + Cz + 8 = 0$. The parametric form of this equation is as shown.
$x = 2 - 3t + s$
$y = -1 + 2t - s$
$z = 3 - t + 2s$

35. What are the values of A and C in the given equation?

A. 6 and −1 **B.** 3 and −8
C. −3 and −4 **D.** −6 and −2

Use the following information to answer the next question.

Jason must convert the scalar form of an equation that defines a plane, $6x - 3y + 7z - 42 = 0$, into an equation in its vector form. His first step is to find the x-, y-, and z-intercepts.

36. After Jason has completed the conversion, which of the following vector equations does **not** represent the plane?

A. $(x, y, z) = (7, 0, 0)+t(7, 14, 0)$ $+s(0, -14, -6)$
B. $(x, y, z) = (0, -14, 0)+t(7, 14, 0)$ $+s(7, 0, -6)$
C. $(x, y, z) = (7, 0, 0)+t(7, 0, -6)$ $+s(-7, -14, 0)$
D. $(x, y, z) = (0, 0, 6)+t(7, 0, -6)$ $+s(0, 14, -6)$

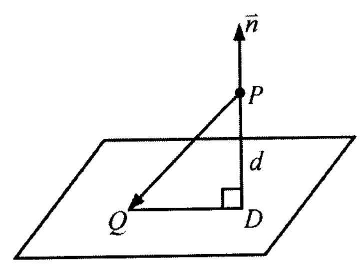
VEC4.7 solve problems relating to lines and planes in three-space that are represented in a variety of ways (e.g., scalar, vector, parametric equations) and involving distances (e.g., between a point and a plane; between two skew lines) or intersections (e.g., of two lines, of a line and a plane), and interpret the result geometrically

Solving Problems Involving Lines and Planes in Three Space

Many problems in two space can be solved by calculating different quantities: the distances between points, the distances between points and lines, the slopes of lines, the midpoints of line segments, and so on. Similar types of problems involving points, lines, and planes can also be solved in three space. In solving these problems, you should be able to visualize the relationships in three space among points, lines, and planes.

The Distance from a Point to a Plane

Visualize a point, P, and a plane, S, in three space. The tip of your pencil (point) held above the surface of your desk (plane) is one possible model.
The distance from the point to the plane is the length of the line segment PD, which is perpendicular to the plane.



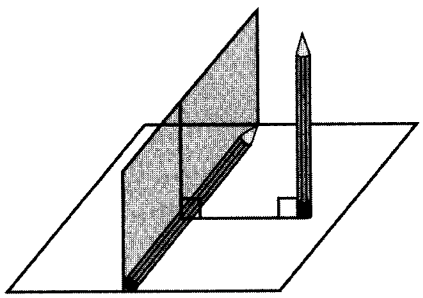
Plane S

If Q is any other point on the plane and $\vec{n}$ is a normal vector to the plane, then distance PD is the magnitude of the projection of vector $\vec{PQ}$ onto $\vec{n}$. If distance PD is represented by d, then

$$d = \frac{|\vec{PQ} \cdot \vec{n}|}{|\vec{n}|}.$$

The Distance between Two Skew Lines

Recall that lines in space do not have a scalar representation but can be represented by vector or parametric equations. Also recall that *skew lines* are non-parallel lines in space that do not intersect. Although skew lines are not parallel, they always lie in parallel planes. You can visualize this relationship by laying a pencil on your desk and then holding a second pencil above the first. You can then take a book or other flat surface and place it against the edge of the first pencil so that it is parallel to the second pencil.



Since the minimum distance between two skew lines is equal to the distance from one of the lines to the parallel plane containing the other line, the approach used for finding the distance between two skew lines can be the same as the one used for finding the distance from a point to a plane. This involves finding a point on each line and the equation of a vector that is normal to the plane.

Intersections of Lines and Intersections of Lines and Planes

There are different configurations resulting from the intersection of lines and planes in three space. Determining the solution to these problems often involves using direction vectors to determine whether lines or planes are parallel and solving systems of equations.

37. To the nearest tenth, the distance between the point $A(8, 2, -5)$ and the plane $4x + 2y - z - 12 = 0$ is

A. 5.7 **B.** 6.3

C. 6.5 **D.** 7.8

Use the following information to answer the next question.

Two lines in three space are given by these equations.
$L_1: (x, y, z) = (3, -7, 5) + s(1, -2, 4)$
$L_2: (x, y, z) = (-7, -8, 4) + t(3, 1, -1)$

Written Response

38. Determine the geometric relationship between the two lines.

Answers and Solutions
Geometry and Algebra of Vectors

1. C	**10. WR**	**b) WR**	**24. B**	**32. WR**
2. B	**11. WR**	**c) WR**	**25. D**	**33. D**
3. C	**12. a) WR**	**17. B**	**26. D**	**34. C**
4. D	**b) WR**	**18. C**	**27. B**	**35. D**
5. B	**13. a) C**	**19. B**	**28. B**	**36. D**
6. B	**b) C**	**20. D**	**29. C**	**37. B**
7. C	**14. 8**	**21. D**	**30. a) WR**	**38. WR**
8. B	**15. WR**	**22. WR**	**b) WR**	
9. 23	**16. a) WR**	**23. C**	**31. B**	

1. C

Vectors must have both magnitude and direction. Weight is the only given option that has both magnitude and direction. The direction is toward the centre of Earth.

Mass, speed, and volume have only magnitude.

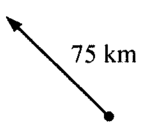
2. B

Scalar quantities have magnitude and are directionless. Temperature is the only given quantity that has magnitude only.

Velocity, displacement, and force have direction as well as magnitude.

3. C

This geometric vector is the only vector (directed line segment) that is pointing in the direction of northwest.

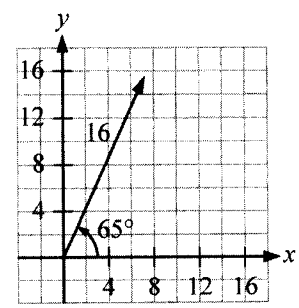


4. D

A vector with Cartesian coordinates of (0, 4) is located 4 units from the origin on the y-axis. Therefore, its length is 4 and the counterclockwise rotation from the positive x-axis is 90°. The polar coordinates of the algebraic vector are (4, 90°).

5. B

Step 1
Sketch the vector.



Step 2
Determine the Cartesian form of the vector.
The representation of the vector using polar coordinates is (16, 65°).
The Cartesian representation is found by using $x = r\cos\theta$ and $y = r\sin\theta$.

$$\begin{aligned} x &= r\cos\theta \\ &= 16\cos 65^\circ \\ &\approx 6.76 \\ y &= r\sin\theta \\ &= 16\sin 65^\circ \\ &\approx 14.50 \end{aligned}$$

Thus, the Cartesian representation of this vector is approximately (6.76, 14.50).

6. B

Use the extension of the Pythagorean theorem to calculate the distance. Let
$A(x_1, y_1, z_1) = A(4, 3, -7)$ and
$B(x_2, y_2, z_2) = B(-2, 4, 2)$.

$$\begin{aligned} AB &= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \sqrt{(-2-4)^2 + (4-3)^2 + (2-(-7))^2} \\ &= \sqrt{(-6)^2 + (1)^2 + (9)^2} \\ &= \sqrt{36 + 1 + 81} \\ &= \sqrt{118} \end{aligned}$$

7. C

Since $v = (3, 7, -2)$ starts at the origin, use the distance formula and the points (0, 0, 0) and (3, 7, −2) to calculate the magnitude.

$$\begin{aligned} |\vec{v}| &= \sqrt{x^2 + y^2 + z^2} \\ &= \sqrt{3^2 + 7^2 + (-2)^2} \\ &= \sqrt{9 + 49 + 4} \\ &= \sqrt{62} \\ &\approx 7.87 \end{aligned}$$

8. B

For any scalar, k, if $\vec{u} = (x_1, y_1, z_1)$, then
$k\vec{u} = (kx_1, ky_1, kz_1)$.

$$\begin{aligned} -\sqrt{8}\vec{u} &= -\sqrt{8}(2, -1, \sqrt{2}) \\ &= -2\sqrt{2}(2, -1, \sqrt{2}) \\ &= \begin{pmatrix} [(-2\sqrt{2})(2)] \\ [(-2\sqrt{2})(-1)] \\ [(-2\sqrt{2})(\sqrt{2})] \end{pmatrix} \\ &= (-4\sqrt{2}, 2\sqrt{2}, -2\sqrt{4}) \\ &= (-4\sqrt{2}, 2\sqrt{2}, -4) \end{aligned}$$

9. 23

Step 1
Simplify the expression $2\vec{p} - 3\vec{q}$.
If $2\vec{p} = 2(-h, 2k)$ and $3\vec{q} = 3(-4h, -3k)$, then
$2\vec{p} - 3\vec{q} = 2(-h, 2k) - 3(-4h, -3k)$.

$$\begin{aligned} 2\vec{p} - 3\vec{q} &= 2(-h, 2k) - 3(-4h, -3k) \\ &= (-2h, 4k) + (12h, 9k) \\ &= (-2h + 12h, 4k + 9k) \\ &= (10h, 13k) \end{aligned}$$

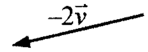
Step 2
Determine the values of a and b.
Since $2\vec{p} - 3\vec{q}$ can be expressed as $(10h, 13k)$, which is equivalent in form to (ah, bk), it follows that $a = 10$ and $b = 13$.

Step 3
Calculate the sum of a and b.

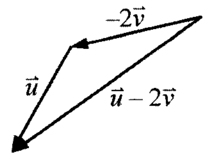
$$\begin{aligned} &a + b \\ &= 10 + 13 \\ &= 23 \end{aligned}$$

10. WR

The vector $-2\vec{v}$ has a magnitude (length) that is twice as great as $\vec{v}$ and is in the opposite direction.



Then, to represent $\vec{u} - 2\vec{v}$, add the vectors $\vec{u}$ and $-2\vec{v}$ geometrically as shown.



11. WR

Evaluate $(\vec{u} - \vec{v}) - \vec{w}$ and $\vec{u} - (\vec{v} - \vec{w})$.

$$\begin{aligned} &(\vec{u} - \vec{v}) - \vec{w} \\ &= ((1, 1, 1) - (-3, -2, 4)) \\ &\quad - (2, 3, -1) \\ &= (1 - (-3), 1 - (-2), 1 - 4) \\ &\quad - (2, 3, -1) \\ &= (4, 3, -3) - (2, 3, -1) \\ &= (4 - 2, 3 - 3, -3 - (-1)) \\ &= (2, 0, -2) \end{aligned}$$

$$\begin{aligned} &\vec{u} - (\vec{v} - \vec{w}) \\ &= (1, 1, 1) - (-3, -2, 4) \\ &\quad - (2, 3, -1) \\ &= (1, 1, 1) \\ &\quad - (-3 - 2, -2 - 3, 4 - (-1)) \\ &= (1, 1, 1) - (-5, -5, 5) \\ &= (1 - (-5), 1 - (-5), 1 - 5) \\ &= (6, 6, -4) \end{aligned}$$

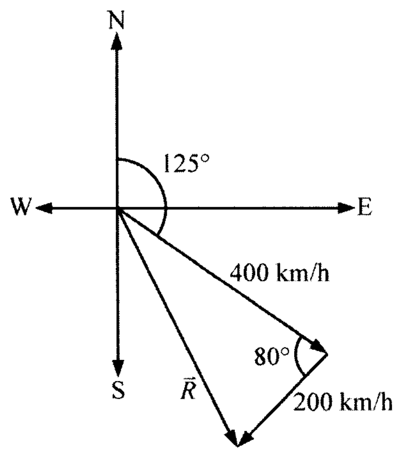
Since $(2, 0, -2) \neq (6, 6, -4)$, the associative property for vector subtraction is **not** true.

12. a) WR

Step 1

Draw a diagram of the two vectors added together indicating the resultant ($\vec{R}$), and the angle between the vectors.

If the angle between the tails of the two vectors is 100°, then the angle between the head and tail of the two vectors is 180° – 100° = 80°.



Step 2

Determine the magnitude of the resultant, $\vec{R}$, using the cosine law.

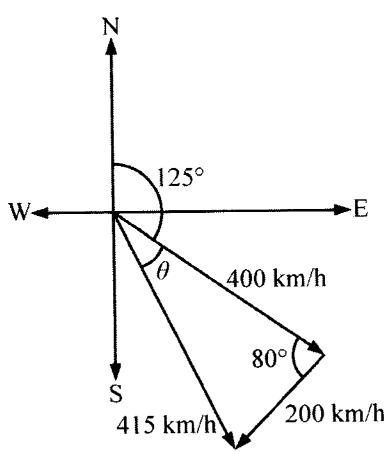
$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$|\vec{R}|^2 = 400^2 + 200^2 - 2(400)(200)\cos 80°$$
$$|\vec{R}|^2 = 172\ 216.2916$$
$$|\vec{R}| = \sqrt{172\ 216.2916}$$
$$\approx 414.9895$$

Therefore, the magnitude of the resultant ground velocity of the airplane is about 415 km/h.

b) WR

Step 1

Draw a diagram of the two vectors and the resultant, indicating all known values and the unknown direction angle, θ.



Step 2

Use a sine law to determine the measure of angle θ.

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$
$$\frac{415}{\sin 80°} = \frac{200}{\sin \theta}$$
$$\sin\theta = \frac{200\sin 80°}{415}$$
$$\sin\theta = 0.474\ 606\ldots$$
$$\theta = \sin^{-1}(0.474\ 606\ldots)$$
$$\theta \approx 28.3°$$

Step 3

Determine the bearing of the airplane's resultant ground velocity.

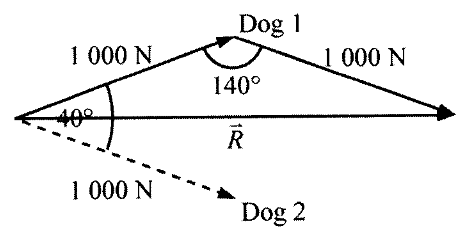
Since the bearing of the airplane (without the wind) is 125° and the direction of the resultant is θ = 28.3°, the bearing of the airplane's resultant ground speed is 125° + 28.3° = 153°, to the nearest whole degree.

13. a) C

Step 1

Draw a vector diagram showing the vector addition and the resultant, $\vec{R}$.

If the angle between the tails of the two vectors is 40°, then the angle between the head and tail of the two vectors is 180° – 40° = 140°.



Step 2

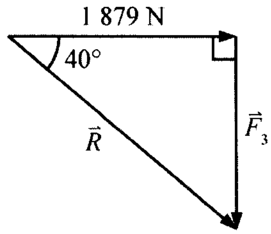
Determine the magnitude of the resultant, $\vec{R}$, using the cosine law.

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$|\vec{R}|^2 = \begin{pmatrix} 1\ 000^2 + 1\ 000^2 \\ - 2(1\ 000)(1\ 000)\cos 140° \end{pmatrix}$$
$$= 3\ 532\ 088.886$$
$$|\vec{R}| = \sqrt{3\ 532\ 088.886}$$
$$\approx 1\ 879.385\ 242$$

To the nearest newton, the magnitude of the resultant force on the sled is 1 879 N.

b) C

Step 1
Draw a diagram that shows the vector addition of the resultant force vector of the two dogs (1 879 N) and the southerly force of the third dog, $\vec{F}_3$.



Step 2
Since the vector diagram forms a right triangle, use the tangent ratio to determine the third dog's force, $\vec{F}_3$.

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$\tan 40° = \frac{|\vec{F}_3|}{1\,879}$$
$$|\vec{F}_3| = 1\,879\tan 40°$$
$$\approx 1\,576.668\,207$$

The southerly force exerted by the third dog would have to be about 1 557 N.

14. 8

Apply the dot product, and solve for b.
If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$, the dot product $\vec{u}\cdot\vec{v}$ is defined as
$\vec{u}\cdot\vec{v} = u_1v_1 + u_2v_2 + u_3v_3$.

$$\vec{u}\cdot\vec{v} = (-3, b, 2)\cdot(5, 2, 7)$$
$$15 = -3(5) + 2b + 2(7)$$
$$15 = -15 + 2b + 14$$
$$15 = -1 + 2b$$
$$16 = 2b$$
$$8 = b$$

The value of b is 8.

15. WR

Step 1
Write equations defining a and b using the definition of orthogonal vectors.
Two non-zero vectors, $\vec{a}$ and $\vec{b}$, are orthogonal if $\vec{a}\cdot\vec{b} = 0$. Therefore,

$$\vec{u}\cdot\vec{v} = 0$$
$$(1, 2, 3)\cdot(a, b, 5) = 0$$
$$1a + 2b + 15 = 0$$
$$① \quad a + 2b = -15$$
$$\vec{u}\cdot\vec{w} = 0$$
$$(1, 2, 3)\cdot(3, a, b) = 0$$
$$3 + 2a + 3b = 0$$
$$② \quad 2a + 3b = -3$$

Step 2
Solve the system of equations using the elimination method.

$① \quad a + 2b = -15$
$② \quad 2a + 3b = -3$

Multiply equation (1) by 2 to form equation (3), and then subtract equation (2).

$③ \quad 2a + 4b = -30$
$② \quad \underline{2a + 3b = -3}$
$\quad\quad b = -27$

Solve for a by substituting the value of b into equation (1).

$$a + 2b = -15$$
$$a + 2(-27) = -15$$
$$a - 54 = -15$$
$$a = 39$$

Therefore, the values of $a = 39$ and $b = -27$ would make the vector $\vec{u}$ orthogonal to both $\vec{v}$ and $\vec{w}$.

16. a) WR

The total magnitude of force acting in the direction of the displacement, $\vec{d}$, is found by adding the magnitudes of the projections of vectors $\vec{F}_1$ and $\vec{F}_2$ on $\vec{d}$.

$$\begin{aligned}
&|\text{proj}_{\vec{d}}\vec{F}_1|\\
&= \frac{|\vec{F}_1 \cdot \vec{d}|}{|\vec{d}|}\\
&= \frac{|(200, 500, 450) \cdot (4, 2, 13)|}{|(4, 2, 13)|}\\
&= \frac{|(200)(4) + (500)(2) + (450)(13)|}{\sqrt{4^2 + 2^2 + 13^2}}\\
&= \frac{7\ 650}{\sqrt{189}}\\
&\approx 556.4556 \text{ N}
\end{aligned}$$

$$\begin{aligned}
&|\text{proj}_{\vec{d}}\vec{F}_2|\\
&= \frac{|\vec{F}_2 \cdot \vec{d}|}{|\vec{d}|}\\
&= \frac{|(200, -400, 300) \cdot (4, 2, 13)|}{|(4, 2, 13)|}\\
&= \frac{|(200)(4) + (-400)(2) + (300)(13)|}{\sqrt{4^2 + 2^2 + 13^2}}\\
&= \frac{3\ 900}{\sqrt{189}}\\
&\approx 283.6833 \text{ N}
\end{aligned}$$

Therefore, the total magnitude of force acting in the direction of the displacement, $\vec{d}$, is $556.4556 + 283.6833 \approx 840$ N, rounded to the nearest newton.

b) WR

Work is equal to the dot product of the force and the displacement. The work done by $\vec{F}_1$ on $\vec{d}$ and by $\vec{F}_2$ on $\vec{d}$ can be calculated as follows:

$$\begin{aligned}
W_1 &= \vec{F}_1 \cdot \vec{d}\\
&= (200, 500, 450) \cdot (4, 2, 13)\\
&= (200)(4) + (500)(2) + (450)(13)\\
&= 7\ 650 \text{ J}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \vec{F}_2 \cdot \vec{d}\\
&= (200, -400, 300) \cdot (4, 2, 13)\\
&= (200)(4) + (-400)(2) + (300)(13)\\
&= 3\ 900 \text{ J}
\end{aligned}$$

Therefore, the total work done on the object is $7\ 650 + 3\ 900 = 11\ 550$ J.

Note that the total work could also be found by taking the magnitude of the force and multiplying it by the magnitude of the displacement.

$$\begin{aligned}
W &= |\text{proj}_{\vec{d}}\vec{F}||\vec{d}|\\
&\approx (840)\sqrt{189}\\
&\approx 11\ 548 \text{ J}
\end{aligned}$$

This value approximates the value given by means of the dot product.

c) WR

To calculate the work done against gravity, use only the z-components of $\vec{F}_1$, $\vec{F}_2$, and $\vec{d}$, which are (0, 0, 450), (0, 0, 300), and (0, 0, 13), respectively.

$$\begin{aligned}
W_1 &= \vec{F}_1 \cdot \vec{d}\\
&= (0, 0, 450) \cdot (0, 0, 13)\\
&= (0)(0) + (0)(0) + (450)(13)\\
&= 5\ 850 \text{ J}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \vec{F}_2 \cdot \vec{d}\\
&= (0, 0, 300) \cdot (0, 0, 13)\\
&= (0)(0) + (0)(0) + (300)(13)\\
&= 3\ 900 \text{ J}
\end{aligned}$$

The total work done against gravity is $5\ 850 + 3\ 900 = 9\ 750$ J.

17. B

Use the definition of work as a dot product to determine the work done by the braking force.

$$\begin{aligned}
W &= |\vec{F} \cdot \vec{d}|\\
&= |\vec{F}||\vec{d}|\cos\theta\\
&= (500)(20.0)\cos 160°\\
&\approx -9\ 397 \text{ J}
\end{aligned}$$

To the nearest joule, the work done by the frictional braking force is −9 397 J.

Note that the reason the work is negative is that the frictional force opposes or slows the object's movement. This occurs when the measure of the angle between $\vec{F}$ and $\vec{d}$ is greater than 90°.

18. C

Step 1

Determine $\vec{u} \times \vec{v}$.

$$\begin{aligned}
&\vec{u} \times \vec{v}\\
&= (u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1)\\
&= \begin{pmatrix} [(2)(2) - (1)(1)], \\ [(1)(-4) - (6)(2)], \\ [(6)(1) - (2)(-4)], \end{pmatrix}\\
&= (4 - 1, (-4) - 12, 6 + 8)\\
&= (3, -16, 14)
\end{aligned}$$

Step 2

Determine $|\vec{u}|$, $|\vec{v}|$, and $|\vec{u}\times\vec{v}|$.

$$\begin{aligned}|\vec{u}| &= \sqrt{6^2+2^2+1^2}\\ &= \sqrt{36+4+1}\\ &= \sqrt{41}\end{aligned}$$

$$\begin{aligned}|\vec{v}| &= \sqrt{(-4)^2+1^2+2^2}\\ &= \sqrt{16+1+4}\\ &= \sqrt{21}\end{aligned}$$

$$\begin{aligned}|\vec{u}\times\vec{v}| &= \sqrt{3^2+(-16)^2+14^2}\\ &= \sqrt{9+256+196}\\ &= \sqrt{461}\end{aligned}$$

Step 3

Substitute $\sqrt{41}$ for $|\vec{u}|$, $\sqrt{21}$ for $|\vec{v}|$, and $\sqrt{461}$ for $|\vec{u}\times\vec{v}|$ in the formula $|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin\theta$, and solve for θ.

$$\begin{aligned}|\vec{u}\times\vec{v}| &= |\vec{u}||\vec{v}|\sin\theta\\ \sqrt{461} &= \sqrt{41}\sqrt{21}\sin\theta\\ \sin\theta &= \frac{\sqrt{461}}{\sqrt{861}}\\ \theta &= \sin^{-1}\left(\frac{\sqrt{461}}{\sqrt{861}}\right)\\ \theta &\approx 47.0°\end{aligned}$$

Step 4

Since $\sin(180° - \theta) = \sin\theta$, θ could also be $180° - 47° = 133°$. Verify which measure of θ is correct by using the definition of the dot product, $\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$.

$$\begin{aligned}\cos\theta &= \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\\ \cos\theta &= \frac{(6, 2, 1)\cdot(-4, 1, 2)}{\sqrt{41}\sqrt{21}}\\ \cos\theta &= \frac{(6)(-4)+(2)(1)+(1)(2)}{\sqrt{861}}\\ \cos\theta &= \frac{-20}{\sqrt{861}}\\ \theta &= \cos^{-1}\left(\frac{-20}{\sqrt{861}}\right)\\ \theta &\approx 133°\end{aligned}$$

19. B

Step 1

Determine $\vec{a}\times\vec{b}$.

$$\begin{aligned}&\vec{a}\times\vec{b}\\ &= \begin{pmatrix}a_2b_3 - a_3b_2\\ a_3b_1 - a_1b_3\\ a_1b_2 - a_2b_1\end{pmatrix}\\ &= \begin{pmatrix}(3)(-2)-(-1)(1)\\ (-1)(k)-(h)(-2)\\ (h)(1)-(3)(k)\end{pmatrix}\\ &= (-6+1, -k+2h, h-3k)\\ &= (-5, 2h-k, h-3k)\end{aligned}$$

Step 2

Since $\vec{a}\times\vec{b} = (-5, 7, 1)$, then $(-5, 7, 1) = (-5, 2h-k, h-3k)$.

Set up a system of two equations for h and k.

① $2h - k = 7$
② $h - 3k = 1$

Step 3

Rearrange equation (2) so that h is the only term on the left side. This will make equation (3).

② $h - 3k = 1$

③ $h = 3k + 1$

Step 4

Substitute equation (3) for h in equation (1), and solve for k.

$$\begin{aligned}2h - k &= 7\\ 2(3k+1) - k &= 7\\ 6k + 2 - k &= 7\\ 5k &= 5\\ k &= 1\end{aligned}$$

Step 5

Substitute the value of k into equation (1), and solve for h.

$$\begin{aligned}2h - k &= 7\\ 2h - 1 &= 7\\ 2h &= 8\\ h &= 4\end{aligned}$$

Therefore, for the cross product $\vec{a}\times\vec{b}$ to be $(-5, 7, 1)$, the values of h and k should be $h = 4$ and $k = 1$.

20. D

Analyze $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ for all three sets, and determine whether or not the two are equal.

Step 1

For set I, the comparison of $(\vec{a}\times\vec{b})\times\vec{c}$ and $\vec{a}\times(\vec{b}\times\vec{c})$ is as follows:

$$\begin{aligned}&(\vec{a}\times\vec{b})\times\vec{c}\\ &= [(1, 0, 0)\times(0, 1, 0)]\times(0, 0, 1)\\ &= (0, 0, 1)\times(0, 0, 1)\\ &= (0, 0, 0)\end{aligned}$$

$$\begin{aligned}&\vec{a}\times(\vec{b}\times\vec{c})\\ &= (1, 0, 0)\times[(0, 1, 0)\times(0, 0, 1)]\\ &= (1, 0, 0)\times(1, 0, 0)\\ &= (0, 0, 0)\end{aligned}$$

$(\vec{a}\times\vec{b})\times\vec{c} = \vec{a}\times(\vec{b}\times\vec{c})$

Step 2
For set II, the comparison of $(\vec{a} \times \vec{b}) \times \vec{c}$ with $\vec{a} \times (\vec{b} \times \vec{c})$ is as follows:

$(\vec{a} \times \vec{b}) \times \vec{c}$
$= [(1, 1, 0) \times (1, -1, 0)] \times (0, 0, -1)$
$= (0, 0, -2) \times (0, 0, -1)$
$= (0, 0, 0)$

$\vec{a} \times (\vec{b} \times \vec{c})$
$= (1, 1, 0) \times [(1, -1, 0) \times (0, 0, -1)]$
$= (1, 1, 0) \times (1, 1, 0)$
$= (0, 0, 0)$

$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$

Step 3
For set III, the comparison of $(\vec{a} \times \vec{b}) \times \vec{c}$ with $\vec{a} \times (\vec{b} \times \vec{c})$ is as follows:

$(\vec{a} \times \vec{b}) \times \vec{c}$
$= [(1, 0, 0) \times (1, 1, 0)] \times (1, 1, 1)$
$= (0, 0, 1) \times (1, 1, 1)$
$= (-1, 1, 0)$

$\vec{a} \times (\vec{b} \times \vec{c})$
$= (1, 0, 0) \times [(1, 1, 0) \times (1, 1, 1)]$
$= (1, 0, 0) \times (1, -1, 0)$
$= (0, 0, -1)$

$(\vec{a} \times \vec{b}) \times \vec{c} \neq \vec{a} \times (\vec{b} \times \vec{c})$

Thus, only set III supports Josh's statement that cross products are not associative, namely that $\vec{a} \times (\vec{b} \times \vec{c}) \neq (\vec{a} \times \vec{b}) \times \vec{c}$.

Sets I and II refute this general principle because the three vectors are all perpendicular to one another. When three vectors are all perpendicular, the cross product of any pair produces a vector collinear to the other, thereby creating a cross product equal to the zero vector.

21. D

Determine the volume of the parallelepiped defined by $\vec{a}$, $\vec{b}$, and $\vec{c}$ by using the formula $V = |\vec{c} \cdot (\vec{a} \times \vec{b})|$.

$$\begin{aligned}
V &= |\vec{c} \cdot (\vec{a} \times \vec{b})| \\
&= \left| (3, -2, -2) \cdot \begin{pmatrix} & (2, -5, -1) \\ \times & (4, 0, 1) \end{pmatrix} \right| \\
&= \left| (3, -2, -2) \cdot \begin{pmatrix} [-5 - 0], \\ [-4 - 2], \\ [0 + 20] \end{pmatrix} \right| \\
&= |(3, -2, -2) \cdot (-5, -6, 20)| \\
&= |(3)(-5) + (-2)(-6) + (-2)(20)| \\
&= |-15 + 12 - 40| \\
&= |-43| \\
&= 43 \text{ units}^3
\end{aligned}$$

22. WR

Method 1

Step 1
Rewrite the equation of each line in the slope-intercept form $y = mx + b$.

l_1: $2x + 3y - 18 = 0$
$$3y = -2x + 18$$
$$y = -\frac{2}{3}x + 6$$

l_2: $5x + \frac{15}{2}y - 210 = 0$
$$\frac{15}{2}y = -5x + 210$$
$$y = \frac{-10}{15}x + \frac{420}{15}$$
$$y = \frac{-2}{3}x + 28$$

Step 2
Compare the slopes (m) and y-intercepts (b) of each line to determine the nature of the lines.

l_1: $m = -\frac{2}{3}$
$b = 6$

l_2: $m = -\frac{2}{3}$
$b = 28$

Since the slopes of the two lines are the same but their y-intercepts are different, these two lines are parallel.

Method 2

Step 1
Create a system of equations in the form $ax + by = c$.

l_1: $2x + 3y - 18 = 0$
① $2x + 3y = 18$

l_2: $5x + \frac{15}{2}y - 210 = 0$
② $5x + \frac{15}{2}y = 210$

Step 2
Multiply equation (1) by 5 and equation (2) by 2. Then, subtract equation (1) from equation (2).

② $\times\ 2\ (10x + 15y = 420)$
① $\times\ 5 \underline{-(10x + 15y = 90)}$
$0x + 0y = 330$

Since there are no solutions (x, y) that satisfy the system, the lines are parallel.

23. C

Solve the equations using a system of equations.

$$\begin{array}{r} 4x - y + z - 3 = 0 \\ \underline{+3x + y + z - 3 = 0} \\ 7x + 0y + 2z - 6 = 0 \\ 7x + 2z = 6 \end{array}$$

The result, $7x + 2z = 6$, is a line of intersection.

24. B

Step 1

Solve for D by substituting (2, 0, 0) into the equation.

$$\begin{aligned} Ax + By + Cz + D &= 0 \\ A(2) + B(0) + C(0) + D &= 0 \\ 2A + D &= 0 \\ D &= -2A \end{aligned}$$

Step 2

Substitute the points (0, 4, 1) and (1, 4, 5) into the equation, and create a system of two equations.

$$\begin{aligned} Ax + By + Cz - 2A &= 0 \\ A(0) + B(4) + C(1) - 2A &= 0 \\ 4B + C - 2A &= 0 \\ ① \ -2A + 4B + C &= 0 \\ Ax + By + Cz - 2A &= 0 \\ A(1) + B(4) + C(5) - 2A &= 0 \\ A + 4B + 5C - 2A &= 0 \\ ② \ -A + 4B + 5C &= 0 \end{aligned}$$

Step 3

Subtract equation (2) from equation (1), and solve for C.

$$\begin{array}{rl} ① & -2A + 4B + C = 0 \\ ② & \underline{-A + 4B + 5C = 0} \\ & -A + 0B - 4C = 0 \\ & -4C = A \\ & C = -\frac{1}{4}A \end{array}$$

Step 4

Substitute $-\frac{1}{4}A$ for C into equation (1), and solve for B.

$$\begin{aligned} -2A + 4B - \frac{1}{4}A &= 0 \\ 4B - \frac{9}{4}A &= 0 \\ 4B &= \frac{9}{4}A \\ B &= \frac{9}{16}A \end{aligned}$$

For the plane $Ax + By + Cz + D = 0$, where $A \neq 0$, $B = \frac{9}{16}A$, $C = -\frac{1}{4}A$, and $D = -2A$.

25. D

Using 3-D graphing technology or pieces of cardboard to represent three different non-parallel planes, it is evident that they could intersect at a single point, in a single line, and in three different lines. However, they could not intersect at only two points.

26. D

Step 1

Determine if Jeremy's statement is correct.

To see if Jeremy's statement is correct, determine the scalar form of the line $Ax + By + C = 0$. Since $\vec{r} = (-3, 5) + t(2, -3)$, then $\vec{r} = (-3, 5) + (2t, -3t)$ and $\vec{r} = (-3 + 2t, 5 - 3t)$.

Therefore, the parametric form of this equation is as follows: $x = -3 + 2t$ and $y = 5 - 3t$.

Now, rewrite each part in terms of t.

$$\begin{aligned} x + 3 &= 2t \\ t &= \frac{x+3}{2} \\ y - 5 &= -3t \\ t &= \frac{y-5}{-3} \end{aligned}$$

Equate the expressions that represent t, and rearrange the result to produce the equation of the line in scalar form.

$$\begin{aligned} t &= t \\ \frac{x+3}{2} &= \frac{y-5}{-3} \\ -3(x+3) &= 2(y-5) \\ -3x - 9 &= 2y - 10 \\ -3x - 2y + 1 &= 0 \\ 3x + 2y - 1 &= 0 \end{aligned}$$

Jeremy's statement that describes the scalar form of the equation of the line is correct.

Step 2
Determine if Derek's statement is correct.
To see if Derek's statement is correct, compare the direction vectors of both lines. The direction vector for the original line is $\vec{m}_1 = (2, -3)$, whereas the direction vector for line L is $(2.5, -3.75)$. For the lines to be parallel, their direction vectors should be scalar multiples of one another.

$$(2, -3) = k(2.5, -3.75) = (2.5k, -3.75k)$$

Find the values of k.

$$2 = 2.5k$$
$$\frac{2}{2.5} = k$$
$$k = 0.8$$
$$-3 = -3.75k$$
$$\frac{-3}{-3.75} = k$$
$$k = 0.8$$

Since $k = 0.8$, the direction vectors are scalar multiples of one another, which makes both lines parallel. Therefore, Derek's statement is correct.

Step 3
Determine if Charissa's statement is correct.
To check Charissa's statement, remember that two vectors are perpendicular if their dot product is equal to 0. Find the dot product between the direction vector $\vec{m}_1 = (2, -3)$ and $\vec{v} = (9, -4)$.

$$\vec{m}_1 \cdot \vec{v} = 0$$
$$(2, -3) \cdot (9, -4) = 0$$
$$(2)(9) + (-3)(-4) = 0$$
$$18 + 12 = 0$$
$$30 \neq 0$$

Since the dot product is not zero, Charissa's statement is incorrect.

Step 4
Determine if Emily's statement is correct.
To check Emily's statement, substitute the points into Jeremy's correct scalar equation.

$$(-1, 2) \Rightarrow 3x + 2y - 1 = 0$$
$$3(-1) + 2(2) - 1 = 0$$
$$0 = 0$$
$$(0, 0.5) \Rightarrow 3x + 2y - 1 = 0$$
$$3(0) + 2(0.5) - 1 = 0$$
$$0 = 0$$

Since both points lie on the line, Emily's statement is correct.

27. B

Step 1
Find two points on the line defined by the vector equation.
One point is the position vector (2, 1, 0). Another point can be found by substituting a value for t, such as $t = 1$.

$$\vec{r} = (2, 1, 0) + 1(-3, 0, 1) = (2, 1, 0) + (-3, 0, 1) = (-1, 1, 1)$$

Step 2
Determine whether the two points are in P_1.
Since the two points have y-values of 1, they are solutions of the plane P_1: $y - 1 = 0$. Therefore, this line lies on plane P_1.

Step 3
Determine which P_2 contains the given line.
To determine the other plane that also contains this line, substitute both points into each plane P_2 of the four given pairs.

Pair A:
$$P_2: x - y - 1 = 0$$
$$(2, 1, 0) \Rightarrow (2) - (1) - 1 = 0$$
$$0 = 0$$
$$(-1, 1, 1) \Rightarrow (-1) - (1) - 1 = 0$$
$$-3 \neq 0$$

Pair B:
$$P_2: x + 3z - 2 = 0$$
$$(2, 1, 0) \Rightarrow 2 + 3(0) - 2 = 0$$
$$0 = 0$$
$$(-1, 1, 1) \Rightarrow (-1) + 3(1) - 2 = 0$$
$$0 = 0$$

Pair C:
$$P_2: 3x + y - 7 = 0$$
$$(2, 1, 0) \Rightarrow 3(2) + 1 - 7 = 0$$
$$0 = 0$$
$$(-1, 1, 1) \Rightarrow 3(-1) + 1 - 7 = 0$$
$$-9 \neq 0$$

Pair D:
$$P_2: -2x + 4y + z = 0$$
$$(2, 1, 0) \Rightarrow -2(2) + 4(1) + 0 = 0$$
$$0 = 0$$
$$(-1, 1, 1) \Rightarrow -2(-1) + 4(1) + 1 = 0$$
$$7 \neq 0$$

Since both points lie on the plane P_2: $x + 3z - 2 = 0$, it is evident that the line is also defined by the intersection of the two planes given in pair B.

28. B

Step 1

Determine a vector form of the equation of the line that passes through the points $P(6, -2, 4)$ and $Q(-1, -4, 0)$.

A direction vector could be as follows:

$$\overrightarrow{PQ} = (-1-6, -4-(-2), 0-4) \\ = (-7, -2, -4)$$

A position vector could be $(6, -2, 4)$, so a vector equation that defines the line could have the form $\vec{r} = (6, -2, 4) + t(-7, -2, -4)$.

Step 2

To establish which of the given points lies on this line, convert the vector equation of the line into its parametric equation form, where $x = a + dt$, $y = b + et$, and $z = c + ft$, and solve for t.

For point $(-8, -6, -4)$, determine the values of t.

$$\vec{r} = (6, -2, 4) + t(-7, -2, -4)$$
$$(-8, -6, -4) = (6, -2, 4) + (-7t, -2t, -4t)$$

$x = a + dt$
$-8 = 6 - 7t$
$t = 2$
$y = b + et$
$-6 = -2 - 2t$
$t = 2$
$z = c + ft$
$-4 = 4 - 4t$
$t = 2$

Since all t-values are $t = 2$, the point $(-8, -6, -4)$ lies on the line.

Step 3

Determine the values of t for the point $\left(\frac{33}{2}, 1, 10\right)$.

$$\vec{r} = (6, -2, 4) + t(-7, -2, -4)$$
$$\left(\frac{33}{2}, 1, 10\right) = (6, -2, 4) + (-7t, -2t, -4t)$$

$x = a + dt$
$\frac{33}{2} = 6 - 7t$
$t = \frac{-3}{2}$
$y = b + et$
$1 = -2 - 2t$
$t = \frac{-3}{2}$
$z = c + ft$
$10 = 4 - 4t$
$t = \frac{-3}{2}$

Since all t-values are $t = \frac{-3}{2}$, the point $\left(\frac{33}{2}, 1, 10\right)$ lies on the line.

Step 4

Determine the values of t for the point $(13, 0, 8)$.

$$\vec{r} = (6, -2, 4) + t(-7, -2, -4)$$
$$(13, 0, 8) = (6, -2, 4) + (-7t, -2t, -4t)$$

$x = a + dt$
$13 = 6 - 7t$
$t = -1$
$y = b + et$
$0 = -2 - 2t$
$t = -1$
$z = c + ft$
$8 = 4 - 4t$
$t = -1$

Since all t-values are $t = -1$, the point $(13, 0, 8)$ lies on the line.

Step 5

Determine the values of t for the point $\left(\frac{23}{2}, -7, -6\right)$.

$$\vec{r} = (6, -2, 4) + t(-7, -2, -4)$$
$$\left(\frac{23}{2}, -7, -6\right) = (6, -2, 4) + (-7t, -2t, -4t)$$

$x = a + dt$
$\frac{23}{2} = 6 - 7t$
$t = -\frac{11}{14}$
$y = b + et$
$-7 = -2 - 2t$
$t = \frac{5}{2}$
$z = c + ft$
$-6 = 4 - 4t$
$t = \frac{5}{2}$

Since all t-values are not equal, the point $\left(\frac{23}{2}, -7, -6\right)$ does not lie on the line.

29. C

For the plane with the equation $Ax + By + Cz + D = 0$, a normal vector is $\vec{n} = (A, B, C)$. Every other vector, $\vec{v}$, that is also normal to the plane is a scalar multiple of $\vec{n}$, namely $\vec{v} = k\vec{n}$, where $k \neq 0$.

For the plane defined by the equation $3x - 4y + z - 12 = 0$, a normal vector is $\vec{n} = (3, -4, 1)$.

Therefore, scalar multiples of $\vec{n} = (3, -4, 1)$ are normal to the plane defined by the scalar equation $3x - 4y + z - 12 = 0$.

Step 1

Determine whether $\vec{u} = \left(-\frac{9}{4}, 3, -\frac{3}{4}\right)$ is a scalar multiple of $\vec{n} = (3, -4, 1)$.

If $\vec{u}$ is a multiple of $\vec{n}$, then there exists a scalar, k, for which the following is true:

$$\vec{u} = k\vec{n}$$
$$\left(-\frac{9}{4}, 3, -\frac{3}{4}\right) = k(3, -4, 1)$$
$$\left(-\frac{9}{4}, 3, -\frac{3}{4}\right) = (3k, -4k, 1k)$$

This implies that $-\frac{9}{4} = 3k$ and $k = -\frac{3}{4}$. It also follows that $3 = -4k$ and $k = -\frac{3}{4}$, and $-\frac{3}{4} = 1k$ and $k = -\frac{3}{4}$.

Since the k-values are all the same, it is clear that $\vec{u} = \left(-\frac{9}{4}, 3, -\frac{3}{4}\right) = -\frac{3}{4}\vec{n}$. So $\vec{u}$ is a scalar multiple of $\vec{n}$.

Therefore, $\vec{u}$ is a vector normal to the plane defined by the scalar equation $3x - 4y + z - 12 = 0$.

Step 2

Determine whether $\vec{v} = (-3, 4, -1)$ is a scalar multiple of $\vec{n} = (3, -4, 1)$.

Without any calculations, it is clear that $\vec{v} = -1\vec{n}$. So, $\vec{v} = (-3, 4, -1)$ is a scalar multiple of $\vec{n} = (3, -4, 1)$.

Therefore, $\vec{v}$ is normal to the plane defined by the scalar equation $3x - 4y + z - 12 = 0$.

Step 3

Determine whether $\vec{w} = \left(-1, \frac{4}{3}, \frac{1}{3}\right)$ is a scalar multiple of $\vec{n} = (3, -4, 1)$.

If $\vec{w}$ is a multiple of $\vec{n}$, then there exists a scalar, k, for which the following is true:

$$\vec{w} = k\vec{n}$$
$$\left(-1, \frac{4}{3}, \frac{1}{3}\right) = k(3, -4, 1)$$
$$\left(-1, \frac{4}{3}, \frac{1}{3}\right) = (3k, -4k, 1k)$$

This implies that $-1 = 3k$ and $k = -\frac{1}{3}$. It also implies that $\frac{4}{3} = -4k$ and $k = -\frac{1}{3}$, and $\frac{1}{3} = 1k$ and $k = \frac{1}{3}$.

Since the k-values are not all the same, vector $\vec{w} = \left(-1, \frac{4}{3}, \frac{1}{3}\right)$ is not a scalar multiple of $\vec{n}$ and is not a vector normal to the plane.

Step 4

Determine whether $\vec{t} = (3, -4, 1)$ is a scalar multiple of $\vec{n} = (3, -4, 1)$.

Since $\vec{t} = 1\vec{n}$, it is clear that $\vec{t} = (3, -4, 1)$ is a scalar multiple of $\vec{n} = (3, -4, 1)$.

Therefore, $\vec{t}$ is normal to the plane defined by the scalar equation $3x - 4y + z - 12 = 0$.

30. a) WR

For a vector, $\vec{u}$, to be parallel to a plane, the dot product between this vector and a normal vector to the plane, $\vec{n}$, must be equal to zero.

$\vec{u} \cdot \vec{n} = 0$

Step 1

Determine a normal of the plane defined by $-2x + 3y - 4z + 12 = 0$.

$\vec{n} = (-2, 3, -4)$.

Step 2

Determine the dot product between $\vec{u} = (5, 6, 2)$ and $\vec{n} = (-2, 3, -4)$.

$$\begin{aligned}\vec{u} \cdot \vec{n} &= (5, 6, 2) \cdot (-2, 3, -4)\\ &= [(5)(-2) + (6)(3) + (2)(-4)]\\ &= -10 + 18 - 8\\ &= 0\end{aligned}$$

Since $\vec{u} \cdot \vec{n} = 0$, vector $\vec{u} = (5, 6, 2)$ is parallel to the plane.

b) WR

Step 1

Find three points that lie on the plane defined by $-2x + 3y - 4z + 12 = 0$. Many different points could be chosen, including $A(2, 0, 2)$, $B(0, -4, 0)$, and $C(1, -2, 1)$.

Step 2

Determine two vectors that lie on the plane.

$$\begin{aligned}\overrightarrow{AB} &= (0, -4, 0) - (2, 0, 2)\\ &= (-2, -4, -2)\\ \overrightarrow{BC} &= (1, -2, 1) - (0, -4, 0)\\ &= (1, 2, 1)\end{aligned}$$

Step 3

Find the resultant, $\overrightarrow{AC}$, by adding the two vectors.

$$\begin{aligned}\overrightarrow{AC} &= \overrightarrow{AB} + \overrightarrow{AC}\\ &= (-2, -4, -2) + (1, 2, 1)\\ &= (-1, -2, -1)\end{aligned}$$

Step 4

Find a normal vector, $\vec{n}$, to the plane.

$\vec{n} = (-2, 3, -4)$

Step 5
For the resultant vector $\overrightarrow{AC}$ to lie on the plane, $\overrightarrow{AC} \cdot \vec{n} = 0$.

$$\begin{aligned}&\overrightarrow{AC} \cdot \vec{n}\\ &= (-1, -2, -1) \cdot (-2, 3, -4)\\ &= [(-1)(-2) + (-2)(3) + (-1)(-4)]\\ &= 2 - 6 + 4\\ &= 0\end{aligned}$$

Since the dot product is equal to zero, the resultant vector $\overrightarrow{AC}$ also lies on the plane defined by $-2x + 3y - 4z + 12 = 0$.

31. B

Step 1
Find the normal vectors, $\vec{n}$, to each plane.

$\vec{n}_1 = (6, -4, 2)$
$\vec{n}_2 = (3, 2, -1)$
$\vec{n}_3 = (9, -6, 3)$

Step 2
Determine whether $\vec{n}_3$ is a scalar multiple of $\vec{n}_1$.

$$\vec{n}_3 = k\vec{n}_1$$
$$(9, -6, 3) = k(6, -4, 2)$$
$$9 = 6k \quad k = \frac{3}{2}$$
$$-6 = -4k \quad k = \frac{3}{2}$$
$$3 = 2k \quad k = \frac{3}{2}$$

The normal $\vec{n}_3$ is a scalar multiple of $\vec{n}_1$. Since $\vec{n}_3 = \frac{3}{2}\vec{n}_1$, planes P_1 and P_3 are parallel.

Step 3
Determine whether $\vec{n}_2$ is a scalar multiple of $\vec{n}_1$.

$$\vec{n}_2 = k\vec{n}_1$$
$$(3, 2, -1) = k(6, -4, 2)$$
$$3 = 6k \quad k = \frac{1}{2}$$
$$2 = -4k \quad k = -\frac{1}{2}$$
$$-1 = 2k \quad k = -\frac{1}{2}$$

Therefore, $\vec{n}_2$ is not a scalar multiple of $\vec{n}_1$.

Since $\vec{n}_2$ is not parallel to $\vec{n}_1$, plane P_3 is distinct.

Thus, two planes are parallel, and the other is distinct.

32. WR

Step 1
Label the system of equations of three planes from 1 to 3.

① $x + 4y + 3z - 5 = 0$
② $x + 3y + 2z - 4 = 0$
③ $x + y - z + 1 = 0$

Step 2
Subtract equation (3) from equation (1) to eliminate x.

① $x + 4y + 3z - 5 = 0$
③ $\underline{x + y - z + 1 = 0}$
④ $3y + 4z - 6 = 0$

Step 3
Subtract equation (3) from equation (2) to eliminate x.

② $x + 3y + 2z - 4 = 0$
③ $\underline{x + y - z + 1 = 0}$
⑤ $2y + 3z - 5 = 0$

Step 4
Subtract 3 times equation (5) from 2 times equation (4) to eliminate y.

2 × ④ $6y + 8z - 12 = 0$
3 × ⑤ $\underline{6y + 9z - 15 = 0}$
$-z + 3 = 0$
$z = 3$

Step 5
Substitute $z = 3$ into equation (4).

$$\begin{aligned}3y + 4z - 6 &= 0\\ 3y + 4(3) - 6 &= 0\\ 3y &= -6\\ y &= -2\end{aligned}$$

Step 6
Substitute $y = -2$ and $z = 3$ into equation (3).

$$\begin{aligned}x + y - z + 1 &= 0\\ x + (-2) - 3 + 1 &= 0\\ x &= 4\end{aligned}$$

Therefore, the solution to the system is a single point $(4, -2, 3)$ and can be written in set notation as $\{(4, -2, 3)\}$.

33. D

Step 1
Determine possible direction vectors of the plane by using points A, B, and C.

$$\begin{aligned}\overrightarrow{AB} &= (-2 - 1, -6 - 12, 4 - 5)\\ &= (-3, -18, -1)\\ \overrightarrow{AC} &= (-1 - 1, -1 - 12, -4 - 5)\\ &= (-2, -13, -9)\\ \overrightarrow{BC} &= (-1 - (-2), -1 - (-6), -4 - 4)\\ &= (1, 5, -8)\end{aligned}$$

Step 2
Determine a vector equation for the plane.

A position vector is any point on the plane, such as point A, B, or C. Therefore, a vector equation of the plane is represented by one of its points and multiples of the sum of two non-parallel direction vectors.

The vector equation

$$(x, y, z) = \begin{pmatrix} (1, 12, 5) \\ + t(-3, -18, -1) \\ + s(-2, -13, -9) \end{pmatrix}$$

correctly defines the plane since its position vector is defined by point A and the direction vectors are defined by $\overrightarrow{AB} = (-3, -18, -1)$ and $\overrightarrow{AC} = (-2, -13, -9)$.

34. C

Step 1
Determine the parametric equation of the plane.

To find a parametric equation of the plane, use points P, Q, and R to determine possible direction vectors.

$$\begin{aligned} \overrightarrow{PQ} &= (-2-4, 0-6, -8-(-2)) \\ &= (-6, -6, -6) \\ \overrightarrow{PR} &= (-10-4, -2-6, 2-(-2)) \\ &= (-14, -8, 4) \\ \overrightarrow{QR} &= (-10-(-2), -2-0, 2-(-8)) \\ &= (-8, -2, 10) \end{aligned}$$

Direction vectors could also be represented by $\overrightarrow{QP} = (6, 6, 6)$, $\overrightarrow{RP} = (14, 8, -4)$, and $\overrightarrow{RQ} = (8, 2, -10)$.

The parametric equation given in alternative A does not represent the plane. It correctly defines the position vector $P(4, 6, -2)$ and the direction vector $\overrightarrow{QP} = (6, 6, 6)$, but it is not defined by the directional vector $\overrightarrow{RQ} = (8, 2, -10)$.

Step 2
Determine the scalar equation of the plane.

To determine the scalar equation $Ax + By + Cz + D = 0$, use two direction vectors, such as $\overrightarrow{PQ} = (-6, -6, -6)$ and $\overrightarrow{PR} = (-14, -8, 4)$, to find a normal vector, $\vec{n}$, to the plane.

$$\begin{aligned} \vec{n} &= \overrightarrow{PQ} \times \overrightarrow{PR} \\ &= (-6, -6, -6) \times (-14, -8, 4) \\ &= [(-6)(4) - (-6)(-8), (-6)(-14) \\ &\quad - (-6)(4), (-6)(-8) - (-6)(-14)] \\ &= (-72, 108, -36) \end{aligned}$$

Since $\vec{n} = (-72, 108, -36)$, the scalar equation can be defined as $-72x + 108y - 36z + D = 0$.

Using one of the points, such as $P(4, 6, -2)$, you can find the value of D.

$$\begin{aligned} -72x + 108y - 36z + D &= 0 \\ -72(4) + 108(6) - 36(-2) + D &= 0 \\ -288 + 648 + 72 + D &= 0 \\ 432 + D &= 0 \\ D &= -432 \end{aligned}$$

Therefore, the scalar equation of the plane is $-72x + 108y - 36z - 432 = 0$ or, in its simplified form, is $2x - 3y + z + 12 = 0$. The scalar equation $2x - 3y + z - 12 = 0$ is incorrect.

Step 3
Determine the x-intercept of the plane.

The x-intercept of the plane can be determined from the scalar equation $2x - 3y + z + 12 = 0$.

Determine the x-intercept as follows:

$$\begin{aligned} 2x - 3y + z + 12 &= 0 \\ 2x - 3(0) + 0 + 12 &= 0 \\ 2x &= -12 \\ x &= -6 \end{aligned}$$

The x-intercept is $(-6, 0, 0)$.

Step 4
Verify if the point $(-2, 1, -7)$ is on the plane using the scalar equation $2x - 3y + z + 12 = 0$.

Substitute the values of the point into the scalar equation, and determine if the LHS = RHS.

LHS	RHS
$2x - 3y + z + 12$	0
$2(-2) - 3(1) + (-7) + 12$	0
$-4 - 3 - 7 + 12$	0
-2	0

Since $-2 \neq 0$, the point $(-2, 1, -7)$ is not a point on the given plane.

35. D

Step 1
Determine the vector equation of the plane.
The equation of the plane can be written in vector form as $(x, y, z) = (2, -1, 3) + t(-3, 2, -1) + s(1, -1, 2)$.
Thus, a point on the plane is $(2, -1, 3)$, and two direction vectors are $\vec{u} = (-3, 2, -1)$ and $\vec{v} = (1, -1, 2)$.

Step 2
Determine a normal vector to the plane.
A vector normal to the plane, $\vec{n}$, can be determined as follows:

$$\begin{aligned}\vec{n} &= \vec{u} \times \vec{v}\\ &= (-3, 2, -1) \times (1, -1, 2)\\ &= [(2)(2) - (-1)(-1), (-1)(1) - (-3)(2), (-3)(-1) - (2)(1)]\\ &= (3, 5, 1)\end{aligned}$$

Thus, the equation of the plane in scalar form is $3x + 5y + z + D = 0$.

Step 3
Substitute the coordinates of the point $(2, -1, 3)$ into the equation to find the value of D.

$$\begin{aligned}3x + 5y + z + D &= 0\\ 3(2) + 5(-1) + 3 + D &= 0\\ 6 - 5 + 3 + D &= 0\\ D &= -4\end{aligned}$$

Therefore, the equation of the plane in scalar form is $3x + 5y + z - 4 = 0$.

Step 4
Rewrite the scalar form of the equation as $Ax - 10y + Cz + 8 = 0$ by multiplying the simplified form by -2.

$$-2(3x + 5y + z - 4) = -2(0)$$
$$-6x - 10y - 2z + 8 = 0$$

Therefore, the values of A and C in the equation are $A = -6$ and $C = -2$.

36. D

Step 1
Determine the x-, y-, and z-intercepts.
The x-intercept can be determined as follows:

$$\begin{aligned}6x - 3y + 7z - 42 &= 0\\ 6x - 3(0) + 7(0) - 42 &= 0\\ 6x &= 42\\ x &= 7\end{aligned}$$

The y-intercept can be determined as follows:

$$\begin{aligned}6x - 3y + 7z - 42 &= 0\\ 6(0) - 3y + 7(0) - 42 &= 0\\ -3y &= 42\\ y &= -14\end{aligned}$$

The z-intercept can be determined as follows:

$$\begin{aligned}6x + 3y + 7z - 42 &= 0\\ 6(0) + 3(0) + 7z - 42 &= 0\\ 7z &= 42\\ z &= 6\end{aligned}$$

The three intercepts are $(7, 0, 0)$, $(0, -14, 0)$, and $(0, 0, 6)$.

Step 2
Determine three direction vectors using the intercept points.

$$\begin{aligned}\vec{u} &= (0, -14, 0) - (7, 0, 0)\\ &= (-7, -14, 0)\\ \vec{v} &= (0, -14, 0) - (0, 0, 6)\\ &= (0, -14, -6)\\ \vec{w} &= (7, 0, 0) - (0, 0, 6)\\ &= (7, 0, -6)\end{aligned}$$

Note that three other direction vectors, $-\vec{u} = (7, 14, 0)$, $-\vec{v} = (0, 14, 6)$, and $-\vec{w} = (-7, 0, 6)$, could also be found by reversing the order of points.

Step 3
Determine the vector equation of the plane.
Vector equations can be formed by using one of the points and two non-parallel direction vectors.
The equation $(x, y, z) = (0, 0, 6) + t(7, 0, -6) + s(0, 14, -6)$ has an incorrect direction vector.
In order for the equation to be correct, $s(0, 14, -6)$ would have to be $s(0, -14, -6)$ or $s(0, 14, 6)$.

37. B

Step 1
Confirm that the point $A(8, 2, -5)$ does not lie on the plane.

$$\begin{aligned}\text{LHS} &= 4(8) + 2(2) - (-5) - 12\\ &= 32 + 4 + 5 - 12\\ &= 29\\ &\neq \text{RHS}\end{aligned}$$

The point $A(8, 2, -5)$ does not lie on the plane.

Step 2
Determine a second point, B, on the plane by selecting arbitrary x-, y-, and z-values.
In this case, $x = 3$ and $y = 0$.

$$\begin{aligned} 4x + 2y - z - 12 &= 0 \\ 4(3) + 2(0) - z - 12 &= 0 \\ 12 - z - 12 &= 0 \\ z &= 0 \end{aligned}$$

Therefore, point $B(3, 0, 0)$ is on the plane.

Step 3
Find the vector $\overrightarrow{AB}$.

$$\begin{aligned} \overrightarrow{AB} &= (3-8, 0-2, 0-(-5)) \\ &= (-5, -2, 5) \end{aligned}$$

Step 4
Determine a normal vector to the plane.
A normal vector, $\vec{n}$, to the plane $4x + 2y - z - 12 = 0$ is $\vec{n} = (4, 2, -1)$.

Step 5
Calculate the distance, d, from point $A(8, 2, -5)$ to the plane by carrying out a projection of $\overrightarrow{AB}$ onto $\vec{n}$.

$$\begin{aligned} d &= \frac{|\overrightarrow{AB} \cdot \vec{n}|}{|\vec{n}|} \\ &= \frac{|(-5, -2, 5) \cdot (4, 2, -1)|}{|(4, 2, -1)|} \\ &= \frac{|-20-4-5|}{\sqrt{4^2 + 2^2 + (-1)^2}} \\ &= \frac{29}{\sqrt{21}} \\ &\approx 6.328\,318\,817 \end{aligned}$$

To the nearest tenth, the distance from point A to the plane is 6.3.

38. WR

Step 1
Determine if the lines are parallel or coincident.
From the direction vectors $(1, -2, 4)$ and $(3, 1, -1)$, it can be seen that the lines are not parallel and also not coincident.

Step 2
Change the equations of the lines into parametric form.

- L_1:
  $x = 3 + s$
  $y = -7 - 2s$
  $z = 5 + 4s$
- L_2:
  $x = -7 + 3t$
  $y = -8 + t$
  $z = 4 - t$

Step 3
Determine if the lines intersect by equating the equations for x, y, and z and then solving the resulting system of equations.

$$\begin{aligned} x_1 &= x_2 \\ 3 + s &= -7 + 3t \\ ① \quad s - 3t &= -10 \\ y_1 &= y_2 \\ -7 - 2s &= -8 + t \\ ② \quad -2s - t &= -1 \\ z_1 &= z_2 \\ 5 + 4s &= 4 - t \\ ③ \quad 4s + t &= -1 \end{aligned}$$

Step 4
Add equations (2) and (3) to solve for s.

$$\begin{aligned} ② \quad -2s - t &= -1 \\ ③ \quad 4s + t &= -1 \\ 2s &= -2 \\ s &= -1 \end{aligned}$$

Step 5
Substitute $s = -1$ into equation (3).

$$\begin{aligned} ③ \quad 4s + t &= -1 \\ 4(-1) + t &= -1 \\ t &= 3 \end{aligned}$$

Step 6
Check to see if $s = -1$ and $t = 3$ also satisfy equation (1).

$$\begin{aligned} ① \quad s - 3t &= -10 \\ (-1) - 3(3) &= -10 \\ -10 &= -10 \end{aligned}$$

Since the values of s and t satisfy all three equations, the two lines must intersect one another.

Step 7
Find the point of intersection by substituting $s = -1$ into the equation defining line L_1.

$$\begin{aligned} L_1&: (x, y, z) \\ &= (3, -7, 5) + s(1, -2, 4) \\ &= (3, -7, 5) - 1(1, -2, 4) \\ &= (3, -7, 5) + (-1, 2, -4) \\ &= (2, -5, 1) \end{aligned}$$

The two lines L_1 and L_2 are intersecting lines with a point of intersection at $(2, -5, 1)$.

UNIT TEST — GEOMETRY AND ALGEBRA OF VECTORS

1. Which of the following measurements is considered a vector quantity?
   A. Sound intensity
   B. Wind velocity
   C. Snow depth
   D. Area

2. Which of the following quantities is a scalar quantity?
   A. Force **B.** Length
   C. Velocity **D.** Acceleration

3. Which of the following compass directions corresponds to a bearing of 315°?
   A. S45°W **B.** N15°E
   C. NW **D.** SE

4. What is the approximate Cartesian form of a vector with a magnitude of 9 and a bearing of 210°?
   A. (3.24, −8.13)
   B. (−4.05, −6.23)
   C. (−4.50, −7.79)
   D. (−2.87, −8.10)

5. The magnitude of vector $\overrightarrow{CD}$, with points $C(-2, -3, 4)$ and $D(4, -2, -6)$, is
   A. $\sqrt{105}$ **B.** $\sqrt{137}$
   C. $\sqrt{161}$ **D.** $\sqrt{184}$

6. If the vectors $\vec{u} = (-1, -4, -5)$, $\vec{s} = (3, -2, 4)$, and $\vec{t} = (-6, 1, 2)$ are put in order of magnitude from least to greatest, the order is
   A. $\vec{s}, \vec{u}, \vec{t}$
   B. $\vec{s}, \vec{t}, \vec{u}$
   C. $\vec{t}, \vec{u}, \vec{s}$
   D. $\vec{t}, \vec{s}, \vec{u}$

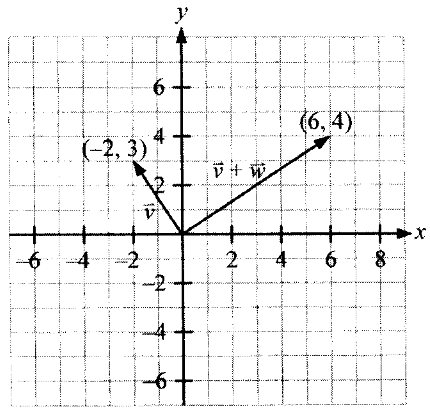
7. If $4.5\vec{u} = \left(\frac{9}{4}, -5, 2\right)$, then $\vec{u}$ is equal to
   A. $\left(\frac{27}{2}, -\frac{45}{2}, \frac{4}{9}\right)$
   B. $\left(\frac{27}{2}, -\frac{45}{2}, 9\right)$
   C. $\left(\frac{1}{2}, -\frac{10}{9}, \frac{4}{9}\right)$
   D. $\left(\frac{1}{2}, -\frac{10}{9}, 9\right)$

Numerical Response

8. If $\vec{u} = (2, 7, -3)$, $\vec{v} = (3, 1, -2)$, and $3\vec{u} - 4\vec{v} = (h, k, l)$, then the value of k is __________.

Use the following information to answer the next question.

Vectors $\vec{v}$ and $\vec{v} + \vec{w}$ are illustrated on the given Cartesian plane.

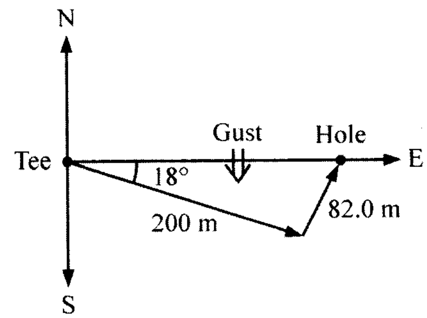


Written Response

9. Sketch the vector $\vec{w}$ on the given grid, and label its coordinates.

Use the following information to answer the next multipart question.

10. Tammy is golfing. She wants to drive her ball from the tee to the hole that lies due east. However, a gust of wind blowing south causes the ball to land 200 m away from the tee, at E18°S. Her second shot is 82.0 m and lands directly in the hole. The given vector diagram illustrates this information.



a) If Tammy's first shot takes 3.25 s to land, the magnitude of the velocity of the gust of wind blowing south is approximately

A. 65.0 m/s **B.** 61.8 m/s

C. 23.0 m/s **D.** 19.0 m/s

Numerical Response

b) To the nearest metre, the distance from the tee to the hole is __________ m.

Use the following information to answer the next question.

Three boys are pulling on an object. Ted and Bill pull on an object with forces of 130 N east and 150 N south, respectively.

11. In which direction should James pull on the object so that it does not move?

A. Northwest **B.** Northeast

C. Southwest **D.** Southeast

12. The magnitude of the projection of $\vec{p} = (7, 3)$ on $\vec{q} = (-4, 2)$ is

A. 22 **B.** −22

C. 4.92 **D.** −4.92

Numerical Response

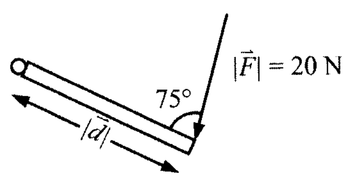
13. The measure of the angle between vectors $\vec{u} = (7, 12, 3)$ and $\vec{v} = (-3, 2, 1)$, to the nearest tenth, is __________°.

Written Response

14. Determine if the dot product is distributive over subtraction $(\vec{a} \cdot (\vec{b} - \vec{c}) = \vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c})$ by using two space Cartesian vectors with variables.

Use the following information to answer the next question.

A force, $|\vec{F}|$, of 20 N is applied to a door with a width, $|\vec{d}|$, in centimetres at 75°, as shown in the given diagram.



15. If the magnitude of the torque, $|\tau|$, is 18 N·m, the width of the door, $|\vec{d}|$, to the nearest centimetre, is

A. 90 cm **B.** 93 cm

C. 111 cm **D.** 115 cm

Use the following information to answer the next multipart question.

16. Two vectors are given as $\vec{u} = (2, -2, 4)$ and $\vec{v} = (-6, -8, 4)$.

Written Response

a) Determine the value of $\vec{u} \times \vec{v}$.

b) Is the vector $\vec{w} = (-6, 8, 7)$ orthogonal to both $\vec{u}$ and $\vec{v}$? Explain your answer.

c) Determine the area of the parallelogram defined by the vectors $\vec{u}$ and $\vec{v}$ to the nearest square unit.

Use the following information to answer the next question.

Maria stated that for any vectors, $\vec{u}$, $\vec{v}$, and $\vec{w}$, and a scalar, $k \in R$, the following properties apply to cross products:

I. $\vec{u} \times \vec{v} = \vec{v} \times \vec{u}$
II. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
III. $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v}$
IV. $\vec{u} \times (k\vec{u}) = \vec{0}$
V. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \times \vec{w}$

17. Which of Maria's statements are **false**?

A. I only

B. V only

C. I and V only

D. II, III, and IV only

Use the following information to answer the next question.

In a particular month, a sports store sold t pairs of basketball shoes at a price of \$$u$ with a 10% discount. It also sold v pairs of cross-training shoes at a price of \$$w$ with a 15% discount.

18. Which of the following expressions describes the revenue made on the shoes during this month?

A. $(t, 0.100u) \cdot (v, 0.15w)$

B. $(t, v) \cdot (0.10u, 0.15w)$

C. $(t, 0.90u) \cdot (v, 0.85w)$

D. $(t, v) \cdot (0.90u, 0.85w)$

Written Response

19. Rounded to the nearest degree, determine the measures of the interior angles of the triangle with vertices $A(-1, 2, 1)$, $B(2, 5, 6)$, and $C(3, 8, -5)$.

Use the following information to answer the next question.

l_1: $2x + y - 8 = 0$
l_2: $Ax + By + 2 = 0$

Written Response

20. Determine possible values for A and B so that the two given lines only intersect at their x-intercept.

21. Which of the following sets of planes in three-dimensional space are parallel?

A. $x = 4$, $y = 4$

B. $x + z = 8$, $y - z = 8$

C. $x + y - z = -1$, $-x - y + z = -1$

D. $3x - 2y + 6z = 0$, $-9x + 6y - 18z = 0$

Written Response

22. Describe the solutions of the equation $2x - 3y = 12$ in three space.

23. A vector form of the equation of the line passing through the points $P(-2, 4)$ and $Q(4, -4)$ is

A. $\vec{r} = (-2, 4) + t(2, 0)$

B. $\vec{r} = (2, 0) + t(-2, 4)$

C. $\vec{r} = (4, -4) + t(3, -4)$

D. $\vec{r} = (4, -4) + t(-8, 6)$

24. Which of the following parametric forms is equivalent to the equation of the line given by $4x + 3y - 6 = 0$?

A. $x = 0 + 3t$, $y = 2 + 4t$

B. $x = 0 + 3t$, $y = 2 - 4t$

C. $x = -2 - 3t$, $y = 3 + 4t$

D. $x = 3 - 3t$, $y = -2 - 4t$

Use the following information to answer the next question.

The vector equation
$\vec{r} = (k, 3, 1) + t(2, 3, l)$, where $k, l \neq 0$, defines a line that passes through the point $P(-3, -2k, -14)$.

25. The value of k in terms of l is

A. $k = \frac{3}{5}l$ **B.** $k = \frac{5}{3}l$

C. $k = -\frac{3}{5}l$ **D.** $k = -\frac{5}{3}l$

Use the following information to answer the next question.

A line in three space passes through points $A(p, q, r)$ and $B(2p, 3q, 4r)$, where $p, q, r \neq 0$.

26. Which of the following parametric equations define the line passing through these two points?

A. $x = p + 3pt$, $y = q + 4qt$, $z = r + 5rt$

B. $x = p + pt$, $y = q + 2qt$, $z = r + 3rt$

C. $x = -p + 3pt$, $y = -q + 4qt$, $z = -r + 5rt$

D. $x = -p - pt$, $y = -q - 2qt$, $z = -r - 3rt$

27. For which of the following sets of non-parallel planes is the intersection a single point?

A. $P_1: 2x - y + 3z - 2 = 0$, $P_2: x - 3y + 2z + 10 = 0$, $P_3: 10x - 10y + 16z + 3 = 0$

B. $P_1: 2x - y + 3z - 2 = 0$, $P_2: x - 3y + 2z + 10 = 0$, $P_3: 10x - 10y + 16z + 6 = 0$

C. $P_1: 2x - y + 3z - 2 = 0$, $P_2: x - 3y + 2z + 10 = 0$, $P_3: 3x + y - z - 4 = 0$

D. $P_1: 2x - y + 3z - 2 = 0$, $P_2: x - 3y + 2z + 10 = 0$, $P_3: 3x + 6y + 3z + 4 = 0$

Use the following information to answer the next question.

A point $P(3, 0, 5)$ lies on a plane that can be defined by the scalar equation $Ax + By + Cz + D = 0$. A normal vector, $\vec{n}$, to this plane is $(2, -3, 1)$.

28. Which of the following points also lies on this plane?

A. $(4, -7, -3)$

B. $(6, 0, -1)$

C. $(2, 3, 2)$

D. $(4, 1, 0)$

Use the following information to answer the next question.

Three planes are given.
$P_1: -x + 3y + z + 10 = 0$
$P_2: x + y + 2z - 8 = 0$
$P_3: 2x - 2y + z - 4 = 0$

29. In terms of their configuration, these three planes are non-parallel and intersect

A. in a plane **B.** at a point

C. at a line **D.** in pairs

Use the following information to answer the next question.

The three planes shown intersect one another.
P_1: $x + 2y + 3z + 4 = 0$
P_2: $x - y - 3z - 8 = 0$
P_3: $x + 5y + 9z + 5 = 0$

Written Response

30. Describe the configuration of the planes by using normals to determine how they intersect.

Use the following information to answer the next question.

A plane contains the line defined by $(x, y, z) = (1, 2, -3) + t(4, 2, -6)$ and is perpendicular to the line defined by $(x, y, z) = (-2, 3, 0) + s(2, 3, -1)$.

31. What is the scalar equation of this plane?

A. $2x + y - 3z + 1 = 0$

B. $2x + 3y - z - 5 = 0$

C. $2x + y - 3z - 13 = 0$

D. $2x + 3y - z - 11 = 0$

Use the following information to answer the next question.

A plane contains the point $A(-5, 2, -4)$ and has two direction vectors, $\vec{AB} = (4, 8, 1)$ and $\vec{AC} = (-3, -2, 0)$.
A student made the following four statements about this plane:

I. The plane has a parametric equation that can be defined as follows:
$x = -5 + 4t - 3s$
$y = 2 + 8t - 2s$
$z = -4 + t$

II. The plane has a vector equation that can be defined as $(x, y, z) = (-5, 2, -4) + t(3, 2, 0) + s(4, 8, 1)$.

III. The point $D(31, 6, 0)$ lies on the plane.

IV. The z-intercept of the plane is $(0, 0, -5)$.

32. Which of the student's statements about the plane is **false**?

A. I **B.** II

C. III **D.** IV

Use the following information to answer the next question.

The scalar form $Ax - 2y + 3z - 6 = 0$ and vector form $(x, y, z) = (0, -3, 0) + t(1, 4, k) + s(0, 3, 2)$ represent the same plane.

33. If $A, k \neq 0$, which of the following relationships between A and k is **true**?

A. $A = 8 - 3k$ **B.** $A = 8 + 3k$

C. $A = 8k + 3$ **D.** $A = 8k - 3$

Use the following information to answer the next question.

The equation of a plane is given by the vector form $(x, y, z) = (3, 1, -2) + t(5, 0, -1) + s(0, 5, 2)$.

34. What is the scalar form representing this plane?

A. $-5x - 10y - 25z - 45 = 0$

B. $-5x + 10y - 25z + 45 = 0$

C. $5x - 10y + 25z - 45 = 0$

D. $5x - 10y + 25z + 45 = 0$

35. How many solutions are there between the plane $P_1: (x, y, z) = (4, -3, -1) + s(1, -3, 1) + t(2, 4 - 3)$ and the line $L_1: (x, y, z) = (3, 1, -2) + k(-1, -1, 1)$?

A. One **B.** Two

C. None **D.** Infinite

Use the following information to answer the next multipart question.

36. Two non-parallel lines are given.

L_1: $x = 2s$
$y = 1 + s$
$z = 2 - 3s$

L_2: $x = 1 - 3t$
$y = -2 - t$
$z = 1 + t$

Written Response

a) Determine if the given lines are skew lines.

b) To the nearest tenth, determine the distance between these lines.

37. If any plane in three space were to intersect two skew lines, it would **not** be possible for the intersection to be at

A. two lines

B. one point

C. two points

D. one point and one line

Answers and Solutions — Unit Test

1. B	10. a) D	c) WR	25. A	34. D
2. B	b) 244	17. C	26. B	35. C
3. C	11. A	18. D	27. C	36. a) WR
4. C	12. C	19. WR	28. B	b) WR
5. B	13. 83.5	20. WR	29. D	37. A
6. B	14. WR	21. C	30. WR	
7. C	15. B	22. WR	31. D	
8. 17	16. a) WR	23. C	32. C	
9. WR	b) WR	24. B	33. A	

1. B

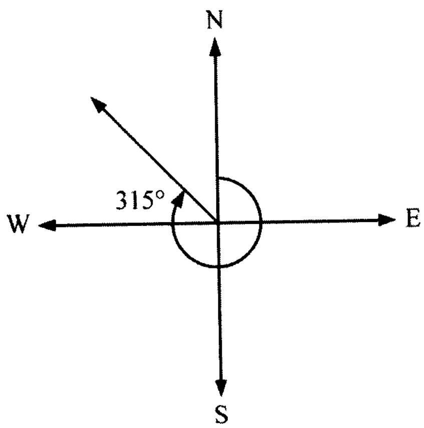
Wind velocity is a measure of velocity. Velocity is the only given measurement that has both magnitude and direction. The other given measurements have only magnitude.

2. B

Scalar quantities have magnitude only. Length is the only given option that has only magnitude. Force, velocity, and acceleration have both magnitude and direction.

3. C

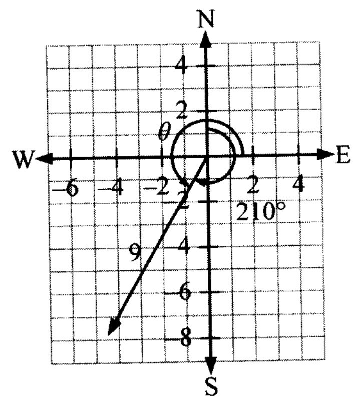
Bearing is measured clockwise from north. Therefore, a bearing of 315° is shown in this diagram.



Since the angle between the vector and due north is 360° – 315° = 45°, the direction is halfway between north and west, that is, NW.

4. C

Step 1
Sketch a diagram of the vector.



Step 2
Change the bearing angle, 210°, to an angle θ with the same terminal arm that is measured in the counterclockwise direction from the positive x-axis (east axis). This angle is $360 - 210 + 90 = 240°$.

Step 3
Determine the Cartesian coordinates.

$$\begin{aligned} x &= r\cos\theta \\ &= 9\cos 240° \\ &\approx -4.50 \end{aligned}$$

$$\begin{aligned} y &= r\sin\theta \\ &= 9\sin 240° \\ &\approx -7.79 \end{aligned}$$

Therefore, the Cartesian representation is (–4.50, –7.79).

5. B

Step 1
Determine the vector $\overrightarrow{CD}$.

$$\begin{aligned} \overrightarrow{CD} &= (x_2 - x_1, y_2 - y_1, z_2 - z_1) \\ &= (4 - (-2), -2 - (-3), -6 - 4) \\ &= (6, 1, -10) \end{aligned}$$

Step 2
Determine the magnitude of vector $\overrightarrow{CD}$ using the formula $|\overrightarrow{CD}| = \sqrt{x^2+y^2+z^2}$.

$$\begin{aligned}|\overrightarrow{CD}| &= \sqrt{x^2+y^2+z^2}\\ &= \sqrt{6^2+1^2+(-10)^2}\\ &= \sqrt{36+1+100}\\ &= \sqrt{137}\end{aligned}$$

6. B

The magnitudes of the vectors are as follows.

$$\begin{aligned}|\vec{u}| &= \sqrt{x^2+y^2+z^2}\\ &= \sqrt{(-1)^2+(-4)^2+(-5)^2}\\ &= \sqrt{1+16+25}\\ &= \sqrt{42}\end{aligned}$$

$$\begin{aligned}|\vec{s}| &= \sqrt{x^2+y^2+z^2}\\ &= \sqrt{3^2+(-2)^2+4^2}\\ &= \sqrt{9+4+16}\\ &= \sqrt{29}\end{aligned}$$

$$\begin{aligned}|\vec{t}| &= \sqrt{x^2+y^2+z^2}\\ &= \sqrt{(-6)^2+1^2+2^2}\\ &= \sqrt{36+1+4}\\ &= \sqrt{41}\end{aligned}$$

Therefore, the order in terms of magnitude from least to greatest is $\vec{s}, \vec{t}, \vec{u}$.

7. C

Since $4.5\vec{u}$ is the same as $\frac{9}{2}\vec{u}$, then

$4.5\vec{u} = \left(\frac{9}{4}, -5, 2\right)$ can be written as

$\frac{9}{2}\vec{u} = \left(\frac{9}{4}, -5, 2\right)$.

Solve the equation $\frac{9}{2}\vec{u} = \left(\frac{9}{4}, -5, 2\right)$ for $\vec{u}$.

$$\frac{9}{2}\vec{u} = \left(\frac{9}{4}, -5, 2\right)$$

$$\frac{\frac{9}{2}\vec{u}}{\frac{9}{2}} = \frac{\left(\frac{9}{4}, -5, 2\right)}{\frac{9}{2}}$$

$$\begin{aligned}\vec{u} &= \frac{2}{9}\left(\frac{9}{4}, -5, 2\right)\\ &= \left(\left(\frac{2}{9}\right)\left(\frac{9}{4}\right), \left(\frac{2}{9}\right)(-5), \left(\frac{2}{9}\right)(2)\right)\\ &= \left(\frac{1}{2}, -\frac{10}{9}, \frac{4}{9}\right)\end{aligned}$$

8. 17

Step 1
Determine $3\vec{u}$.

$$\begin{aligned}3\vec{u} &= 3(2, 7, -3)\\ &= (3(2), 3(7), 3(-3))\\ &= (6, 21, -9)\end{aligned}$$

Step 2
Determine $4\vec{v}$.

$$\begin{aligned}4\vec{v} &= 4(3, 1, -2)\\ &= (4(3), 4(1), 4(-2))\\ &= (12, 4, -8)\end{aligned}$$

Step 3
Determine $3\vec{u} - 4\vec{v}$ by substituting $(6, 21, -9)$ for $3\vec{u}$ and $(12, 4, -8)$ for $4\vec{v}$.

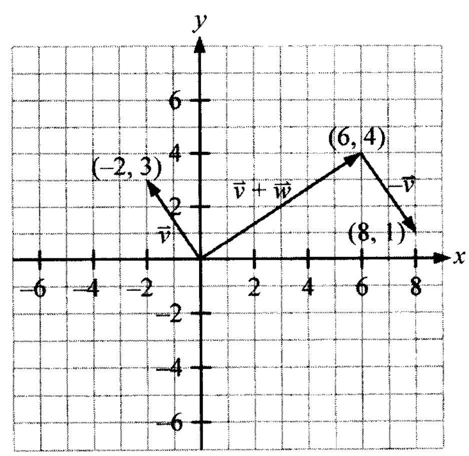
$$\begin{aligned}&3\vec{u} - 4\vec{v}\\ &= (6, 21, -9) - (12, 4, -8)\\ &= (6-12, 21-4, -9-(-8))\\ &= (-6, 17, -1)\end{aligned}$$

Thus, if $(-6, 17, -1) = (h, k, l)$, $k = 17$.

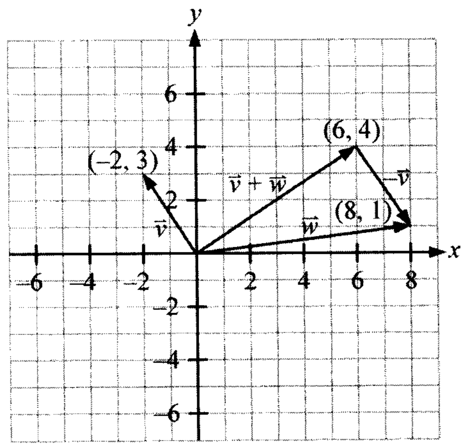
9. WR

Given vectors $\vec{v}$ and $\vec{v} + \vec{w}$, you can find $\vec{w}$ as follows:
$\vec{w} = (\vec{v} + \vec{w}) - \vec{v}$ or $(\vec{v} + \vec{w}) + (-\vec{v})$

Step 1
Draw the geometric addition of vectors $\vec{v} + \vec{w}$ and $(-\vec{v})$ by reversing the direction of vector $\vec{v}$ and joining it's tail to the head of vector $\vec{v} + \vec{w}$.



Step 2
Determine the coordinates of vector $\vec{w}$ by joining the initial point of vector $\vec{v} + \vec{w}$ to the terminal point of vector $-\vec{v}$.



Therefore, the coordinates of vector $\vec{w}$ are (8, 1).

Step 3
Verify the coordinates of vector $\vec{w}$ by subtracting the Cartesian coordinates of $\vec{v}$ from the Cartesian coordinates of $\vec{v} + \vec{w}$.

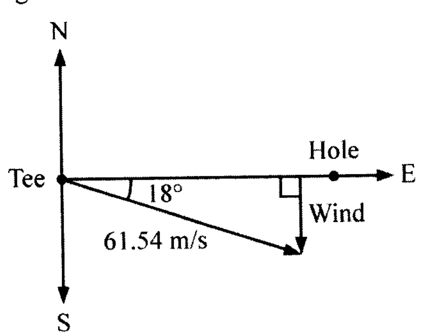
$$\begin{aligned}\vec{w} &= (\vec{v} + \vec{w}) - \vec{v}\\ &= (6, 4) - (-2, 3)\\ &= (6 - (-2), 4 - 3)\\ &= (8, 1)\end{aligned}$$

10. a) D

Step 1
Determine the magnitude of the average horizontal velocity of the ball during the first shot.

$$\begin{aligned}|\vec{v}| &= \frac{|\vec{d}|}{t}\\ &= \frac{200 \text{ m}}{3.25 \text{ s}}\\ &\approx 61.54 \text{ m/s}\end{aligned}$$

Step 2
Draw a vector diagram illustrating the velocity of the ball and the velocity of the gust of wind blowing south.



Note that it is evident that the triangle formed is a right triangle.

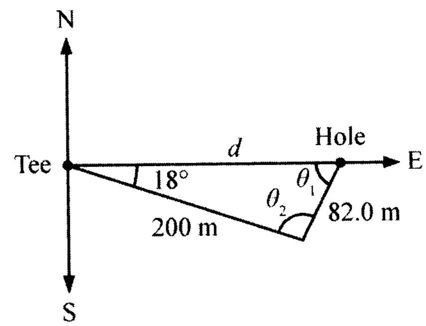
Step 3
Use the sine trigonometric ratio to determine the magnitude of the wind's velocity, w.

$$\begin{aligned}\sin\theta &= \frac{\text{opp}}{\text{hyp}}\\ \sin 18^\circ &= \frac{w}{61.54}\\ w &= 61.54\sin 18^\circ\\ &\approx 19.0 \text{ m/s}\end{aligned}$$

Therefore, the magnitude of the velocity of the gust of wind is approximately 19.0 m/s.

b) 244

Step 1
Draw a diagram of the original triangle, labelling the given sides and angle and the missing angles and side as θ_1, θ_2, and d, respectively.



Step 2
Determine the measure of angle θ_1 using the sine law.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{b}{\sin B}\\ \frac{82.0}{\sin 18^\circ} &= \frac{200}{\sin\theta_1}\\ \sin\theta_1 &= \frac{200\sin 18^\circ}{82.0}\\ \sin\theta_1 &= 0.753\,6999\ldots\\ \theta_1 &= \sin^{-1}(0.753\,6999\ldots)\\ \theta_1 &\approx 48.9119^\circ\end{aligned}$$

Step 3
Determine the measure of angle θ_2.

$$\begin{aligned}\theta_2 &\approx 180^\circ - 18^\circ - 48.9119^\circ\\ &\approx 113.0881^\circ\end{aligned}$$

Step 4

Determine the measure of d using the cosine law.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$d^2 = \begin{pmatrix} 200^2 + 82.0^2 \\ -\ 2(200)(82.0)\cos 113.0881° \end{pmatrix}$$

$$d^2 \approx 59\ 586.39$$

$$d \approx \sqrt{59\ 586.39}$$

$$d \approx 244.1 \text{ m}$$

Note that the sine law could also have been used to find d.

Therefore, to the nearest metre, the distance from the tee to the hole is 244 m.

11. A

Step 1

Determine the approximate direction of the resultant, R, between the two given forces.

Since Ted pulls east and Bill pulls south (with approximately the same force), the resultant force would be a vector in the southeast direction.

Step 2

Determine the approximate direction in which James should pull on the object.

In order for the object not to move, James would have to pull with a force in the opposite direction of the resultant force. This force is called the equilibrant. In this case, since the resultant is in a southeast direction, the equilibrant force that James needs to apply to the object would need to be in the northwest direction.

12. C

Determine the magnitude of the projection of $\vec{p}$ on $\vec{q}$ using the definition of projection.

$$\begin{aligned} |\text{proj}_{\vec{q}}\vec{p}| &= \frac{|\vec{p} \cdot \vec{q}|}{|\vec{q}|} \\ &= \frac{|(7, 3) \cdot (-4, 2)|}{\sqrt{(-4)^2 + 2^2}} \\ &= \frac{|7(-4) + 3(2)|}{\sqrt{20}} \\ &= \frac{22}{\sqrt{20}} \\ &\approx 4.919\ 349\ 55 \end{aligned}$$

Therefore, the magnitude of the projection of $\vec{p}$ on $\vec{q}$ is 4.92.

13. 83.5

Apply the definition of the dot product to determine the measure of the angle, θ, between the vectors.

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|\cos\theta$$

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

$$\cos\theta = \frac{(7, 12, 3) \cdot (-3, 2, 1)}{\left(\sqrt{7^2 + 12^2 + 3^2}\right)\left(\sqrt{(-3)^2 + 2^2 + 1^2}\right)}$$

$$\cos\theta = \frac{7(-3) + 12(2) + 3(1)}{(\sqrt{202})(\sqrt{14})}$$

$$\cos\theta = \frac{6}{53.1789\ldots}$$

$$\cos\theta = 0.112\ 8266\ldots$$

$$\theta = \cos^{-1}(0.112\ 8266\ldots)$$

$$\theta \approx 83.5217°$$

The measure of the angle between the vectors, to the nearest tenth, is 83.5°.

14. WR

Let vector $\vec{a} = (m, n)$, $\vec{b} = (p, q)$, and $\vec{c} = (r, s)$.

Now, apply these vectors to each side of the property.

$$\begin{aligned} &\vec{a} \cdot (\vec{b} - \vec{c}) \\ &= (m, n) \cdot [(p, q) - (r, s)] \\ &= (m, n) \cdot (p - r, q - s) \\ &= m(p - r) + n(q - s) \\ &= mp - mr + nq - ns \end{aligned}$$

$$\begin{aligned} &\vec{a} \cdot \vec{b} - \vec{a} \cdot \vec{c} \\ &= (m, n) \cdot (p, q) - (m, n) \cdot (r, s) \\ &= mp + nq - (mr + ns) \\ &= mp + nq - mr - ns \\ &= mp - mr + nq - ns \end{aligned}$$

Since the left side of the property is equivalent to the right side, the dot product is distributive over subtraction.

15. B

The magnitude of torque is the cross product of the applied force vector and the lever arm vector.

$$|\vec{\tau}| = |\vec{d} \times \vec{F}|$$

$$|\vec{\tau}| = |\vec{d}||\vec{F}|\sin\theta$$

Rewrite the equation in terms of $|\vec{d}|$, and solve.

$$\begin{aligned} |\tau| &= |\vec{d}||\vec{F}|\sin\theta \\ |\vec{d}| &= \frac{|\vec{\tau}|}{|\vec{F}|\sin\theta} \\ &= \frac{18 \text{ N}\cdot\text{m}}{20\text{N}(\sin 75°)} \\ &\approx 0.931\ 75 \text{ m} \\ &\approx 93.175 \text{ cm} \end{aligned}$$

The width of the door, to the nearest centimetre, is 93 cm.

16. a) WR

$\vec{u} \times \vec{v}$

$$= \begin{pmatrix} [u_2v_3 - u_3v_2], \\ [u_3v_1 - u_1v_3], \\ [u_1v_2 - u_2v_1] \end{pmatrix}$$

$$= \begin{pmatrix} [(-2)(4) - (4)(-8)], \\ [(4)(-6) - (2)(4)], \\ [(2)(-8) - (-2)(-6)] \end{pmatrix}$$

$= ([-8 + 32], [-24 - 8], [-16 - 12])$

$= (24, -32, -28)$

b) WR

Method 1

Vector $\vec{w} = (-6, 8, 7)$ is orthogonal to vectors $\vec{u}$ and $\vec{v}$ if $\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$.

$\vec{u} \cdot \vec{w} = (2, -2, 4) \cdot (-6, 8, 7)$

$= (2)(-6) + (-2)(8) + (4)(7)$

$= -12 - 16 + 28$

$= 0$

$\vec{v} \cdot \vec{w} = (-6, -8, 4) \cdot (-6, 8, 7)$

$= 36 - 64 + 28$

$= 0$

Since $\vec{u} \cdot \vec{w} = 0$ and $\vec{v} \cdot \vec{w} = 0$, vector $\vec{w} = (-6, 8, 7)$ is orthogonal to both vectors $\vec{u}$ and $\vec{v}$.

Method 2

By definition, the cross product $\vec{u} \times \vec{v}$ describes a vector orthogonal to both $\vec{u}$ and $\vec{v}$. Therefore, show that vector $\vec{w}$ is collinear with $\vec{u} \times \vec{v}$.

$\vec{w} = k(\vec{u} \times \vec{v})$

$(-6, 8, 7) = k(24, -32, -28)$

$(-6, 8, 7) = (24k, -32k, -28k)$

$24k = -6$

$k = -\frac{1}{4}$

$-32k = 8$

$k = -\frac{1}{4}$

$-28k = 7$

$k = -\frac{1}{4}$

Since $k = -\frac{1}{4}$ for each vector component and $\vec{w} = -\frac{1}{4}(\vec{u} \times \vec{v})$, $\vec{w}$ is collinear with $\vec{u} \times \vec{v}$.

c) WR

The area of the parallelogram formed by vectors $\vec{u}$ and $\vec{v}$ is equal to $|\vec{u} \times \vec{v}|$.

$|\vec{u} \times \vec{v}| = |(24, -32, -28)|$

$= \sqrt{24^2 + (-32)^2 + (-28)^2}$

$= \sqrt{576 + 1\,024 + 784}$

$= \sqrt{2\,384}$

$\approx 48.826\,222\,46$

To the nearest square unit, the area of the parallelogram is 49 units^2.

17. C

Let $\vec{u} = (1, 0, 0)$, $\vec{v} = (1, 1, 0)$, and $\vec{w} = (0, 1, 1)$ and a scalar of $k = 2$.

Step 1

Check statement I.

Since cross products are not commutative, statement I is false.

$\vec{u} \times \vec{v}$

$= (1, 0, 0) \times (1, 1, 0)$

$= (0, 0, 1)$

$\vec{v} \times \vec{u}$

$= (1, 1, 0) \times (1, 0, 0)$

$= (0, 0, -1)$

$\vec{u} \times \vec{v} \neq \vec{v} \times \vec{u}$

The true statement would be $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.

Step 2

Check statement II.

Since cross products are distributive over addition, statement II is true.

$\vec{u} \times (\vec{v} + \vec{w})$

$= (1, 0, 0) \times [(1, 1, 0) + (0, 1, 1)]$

$= (1, 0, 0) \times (1, 2, 1)$

$= (0, -1, 2)$

Thus, $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

Step 3

Check statement III.

Since scalar multiplication of a cross product is commutative, statement III is true.

$k(\vec{u} \times \vec{v})$

$= 2[(1, 0, 0) \times (1, 1, 0)]$

$= 2(0, 0, 1)$

$= (0, 0, 2)$

$(k\vec{u}) \times \vec{v}$

$[2(1, 0, 0)] \times (1, 1, 0)$

$= (2, 0, 0) \times (1, 1, 0)$

$= (0, 0, 2)$

Thus, $k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v}$.

Step 4

Check statement IV.

Since cross products of collinear vectors results in the zero vector, statement IV is true.

$\vec{u} \times k\vec{u}$
$= (1, 0, 0) \times [2(1, 0, 0)]$
$= (1, 0, 0) \times (2, 0, 0)$
$= (0, 0, 0)$
$= \vec{0}$

$\vec{u} \times k\vec{u} = \vec{0}$

Step 5

Check statement V.

Since cross products are not associative, statement V is false.

$\vec{u} \times (\vec{v} \times \vec{w})$
$= (1, 0, 0) \times [(1, 1, 0) \times (0, 1, 1)]$
$= (1, 0, 0) \times (1, -1, 1)$
$= (0, -1, -1)$

$(\vec{u} \times \vec{v}) \times \vec{w}$
$= [(1, 0, 0) \times (1, 1, 0)] \times (0, 1, 1)$
$= (0, 0, 1) \times (0, 1, 1)$
$= (-1, 0, 0)$

Thus, $\vec{u} \times (\vec{v} \times \vec{w}), \neq (\vec{u} \times \vec{v}) \times \vec{w}$.

18. D

Step 1

Write an expression that represents the price of the basketball shoes.

$u - 10\%(u) = u - 0.10u$
$= 0.90u$

Step 2

Write an expression that represents the price of the cross-training shoes.

$w - 15\%(w) = w - 0.15w$
$= 0.85w$

Step 3

Identify the vectors for the number of shoes sold and the price of the shoes.

The vector representing the number of shoes sold is (t, v), and the vector representing the price of the shoes is $(0.90u, 0.85w)$.

Step 4

Take the dot product of the vectors to determine the expression for the revenue made.

The revenue earned for the month would be represented by the dot product $(t, v) \cdot (0.90u, 0.85w)$.

19. WR

Step 1

Determine the vectors $\vec{AB}$ and $\vec{AC}$.

$\vec{AB} = (2, 5, 6) - (-1, 2, 1)$
$= (3, 3, 5)$
$\vec{AC} = (3, 8, -5) - (-1, 2, 1)$
$= (4, 6, -6)$

Step 2

Find the measure of the angle between vectors $\vec{AB}$ and $\vec{AC}$.

$$\begin{aligned}\cos\theta &= \frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}||\vec{AC}|}\\ &= \frac{(3, 3, 5) \cdot (4, 6, -6)}{\sqrt{3^2+3^2+5^2}\sqrt{4^2+6^2+(-6)^2}}\\ &= \frac{12+18-30}{\sqrt{43}\sqrt{88}}\\ &= \frac{0}{\sqrt{3\,784}}\\ &= 0\end{aligned}$$

When $\cos\theta = 0$, the measure of the angle $\theta = 90°$. Therefore, the measure of the interior angle between vectors $\vec{AB}$ and $\vec{AC}$ is 90°.

Step 3

Determine the vectors $\vec{BC}$ and $\vec{BA}$.

$\vec{BC} = (3, 8, -5) - (2, 5, 6)$
$= (1, 3, -11)$
$\vec{BA} = (-1, 2, 1) - (2, 5, 6)$
$= (-3, -3, -5)$

Step 4

Find the measure of the angle between vectors $\vec{BC}$ and $\vec{BA}$.

$$\begin{aligned}\cos\theta &= \frac{\vec{BC} \cdot \vec{BA}}{|\vec{BC}||\vec{BA}|}\\ &= \frac{(1, 3, -11) \cdot (-3, -3, -5)}{\sqrt{1^2+3^2+(-11)^2}\sqrt{(-3)^2+(-3)^2+(-5)^2}}\\ &= \frac{-3-9+55}{\sqrt{131}\sqrt{43}}\\ &= \frac{43}{\sqrt{5\,633}}\\ &\approx 0.572\,9261\\ \theta &\approx \cos^{-1}(0.572\,9261)\\ &\approx 55.0455°\end{aligned}$$

The measure of the interior angle between vectors $\vec{BC}$ and $\vec{BA}$ is 55°.

Step 5
Determine the measure of the angle between vectors $\overrightarrow{CB}$ and $\overrightarrow{CA}$.
Since the sum of the measures of the interior angles of a triangle is 180°, the measure of the other interior angle between vectors $\overrightarrow{CB}$ and $\overrightarrow{CA}$ is
$180° - 90° - 55° = 35°$.
Note that the measures of the interior angles could also have been found using the formula
$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$.

20. WR

Step 1
Find the x-intercept for line l_1 by setting $y = 0$.

$$\begin{aligned} l_1: 2x + y - 8 &= 0 \\ 2x + 0 - 8 &= 0 \\ 2x &= 8 \\ x &= 4 \end{aligned}$$

The x-intercept for line l_1 is (4, 0). This must also be the x-intercept for line l_2, since this is where the two lines intersect one another.

Step 2
Substitute 4 for x and 0 for y into l_2, and solve for A.

$$\begin{aligned} Ax + By + 2 &= 0 \\ A(4) + B(0) + 2 &= 0 \\ 4A &= -2 \\ A &= -\frac{1}{2} \end{aligned}$$

Step 3
Rewrite each line in the slope-intercept form $y = mx + b$.
Note that B can be any value except the one that would result in line l_2 having the same slope as that of line l_1. If they had the same slope, the lines would not intersect, but instead be coincident.

$$\begin{aligned} l_1: 2x + y - 8 &= 0 \\ y &= -2x + 8 \end{aligned}$$

$$\begin{aligned} l_2: -\frac{1}{2}x + By + 2 &= 0 \\ By &= \frac{1}{2}x - 2 \\ y &= \frac{1}{2B}x - \frac{2}{B} \end{aligned}$$

Step 4
Determine the restriction on B.

$$\begin{aligned} -2 &\neq \frac{1}{2B} \\ -4B &\neq 1 \\ B &\neq -\frac{1}{4} \end{aligned}$$

For the two lines to only intersect at their x-intercept, the value of A must be $-\frac{1}{2}$ and B must be any real number except $-\frac{1}{4}$.

21. C

In set A, the plane defined by $x = 4$ is parallel to the y-z plane, and the plane defined by $y = 4$ is parallel to the x-z plane. Therefore, these two planes would intersect at a line.

In set B, the plane defined by $x + z = 8$ is parallel to the y-axis, and the plane defined by $y - z = 8$ is parallel to the x-axis. Therefore, these two planes intersect at a line.

In set C, the two planes defined by $x + y - z = -1$ and $-x - y + z = -1$ are parallel and have no solutions. This can be shown by solving the system.

$$\begin{aligned} ① \quad & x + y - z = -1 \\ + ② \quad & \underline{-x - y + z = -1} \\ & 0x + 0y + 0z = -2 \end{aligned}$$

The resulting equation has no solutions (x, y, z).

In set D, the two planes defined by $3x - 2y + 6z = 0$ and $-9x + 6y - 18z = 0$ are scalar multiples of each other. Therefore, they are coincident. The equations are equivalent and have an infinite number of solutions.

22. WR

In three space, the equation $2x - 3y = 12$ when graphed defines a plane parallel to the z-axis, since the z-coefficient is 0 $(2x - 3y + 0z = 12)$.

Step 1
Determine the x-intercept.

$$\begin{aligned} 2x - 3y &= 12 \\ 2x - 3(0) &= 12 \\ 2x &= 12 \\ x &= 6 \end{aligned}$$

Step 2
Determine the y-intercept.

$$\begin{aligned} 2x - 3y &= 12 \\ 2(0) - 3y &= 12 \\ -3y &= 12 \\ y &= -4 \end{aligned}$$

The plane will have an x-intercept of 6 and a y-intercept of -4. It will contain all lines parallel to the line $2x - 3y = 12$ along the z-axis.

23. C

Step 1
Find the slope of the line passing through $P(-2, 4)$ and $Q(4, -4)$.

$$\begin{aligned} m &= \frac{-4-4}{4-(-2)} \\ &= \frac{-8}{6} \\ &= -\frac{4}{3} \end{aligned}$$

Step 2
The slope of the line is defined as $m = \frac{\Delta y}{\Delta x}$ for a direction vector $\vec{m} = (x, y)$. Determine the slopes of the lines for the given equations.

Alternative A: $\vec{m} = (2, 0)$

$$\begin{aligned} m &= \frac{0}{2} \\ &= 0 \end{aligned}$$

Alternative B: $\vec{m} = (-2, 4)$

$$\begin{aligned} m &= \frac{4}{-2} \\ &= -2 \end{aligned}$$

Alternative C: $\vec{m} = (3, -4)$

$$\begin{aligned} m &= \frac{-4}{3} \\ &= -\frac{4}{3} \end{aligned}$$

Alternative D: $\vec{m} = (-8, 6)$

$$\begin{aligned} m &= \frac{6}{-8} \\ &= -\frac{3}{4} \end{aligned}$$

The equation $\vec{r} = (4, -4) + t(3, -4)$ has the correct direction vector defining the slope of the line. Since $(4, -4)$ is a possible position vector, the form $\vec{r} = (4, -4) + t(3, -4)$ correctly defines the line passing through points $P(-2, 4)$ and $Q(4, -4)$.

24. B

Step 1
To determine which parametric form is equivalent to the given equation of the line, set $t = 1$ and find the corresponding points (x, y).

Find the corresponding point of $x = 0 + 3t$ and $y = 2 + 4t$.

$$\begin{aligned} x &= 0 + 3(1) \\ &= 3 \\ y &= 2 + 4(1) \\ &= 6 \end{aligned}$$

$(x, y) = (3, 6)$

Find the corresponding point of $x = 0 + 3t$ and $y = 2 - 4t$.

$$\begin{aligned} x &= 0 + 3(1) \\ &= 3 \\ y &= 2 - 4(1) \\ &= -2 \end{aligned}$$

$(x, y) = (3, -2)$

Find the corresponding point of $x = -2 - 3t$ and $y = 3 + 4t$.

$$\begin{aligned} x &= -2 - 3(1) \\ &= -5 \\ y &= 3 + 4(1) \\ &= 7 \end{aligned}$$

$(x, y) = (-5, 7)$

Find the corresponding point of $x = 3 - 3t$ and $y = -2 - 4t$.

$$\begin{aligned} x &= 3 - 3(1) \\ &= 0 \\ y &= -2 - 4(1) \\ &= -6 \end{aligned}$$

$(x, y) = (0, -6)$

Step 2
Determine which of these points lies on the line by substituting the coordinates into $4x + 3y - 6 = 0$.

Substitute the coordinates $(3, 6)$.

$$\begin{aligned} 4(3) + 3(6) - 6 &= 0 \\ 24 &\neq 0 \end{aligned}$$

Substitute the coordinates $(3, -2)$.

$$\begin{aligned} 4(3) + 3(-2) - 6 &= 0 \\ 0 &= 0 \end{aligned}$$

Substitute the coordinates $(-5, 7)$.

$$\begin{aligned} 4(-5) + 3(7) - 6 &= 0 \\ -5 &\neq 0 \end{aligned}$$

Substitute the coordinates $(0, -6)$.

$$\begin{aligned} 4(0) + 3(-6) - 6 &= 0 \\ -24 &\neq 0 \end{aligned}$$

The correct alternative is $x = 0 + 3t$ and $y = 2 - 4t$.

25. A

Step 1
Rewrite the equation $\vec{r} = (k, 3, 1) + t(2, 3, l)$ from vector form to parametric form.

$x = k + 2t$, $y = 3 + 3t$, $z = 1 + lt$.

Step 2
Substitute the coordinates of the point $P(-3, -2k, -14)$ for x, y, and z.

$x = k + 2t$
① $-3 = k + 2t$
$y = 3 + 3t$
② $-2k = 3 + 3t$
$z = 1 + lt$
③ $-14 = 1 + lt$

Step 3
Isolate the variable t in equation (3).

$-14 = 1 + lt$
$-15 = lt$
$t = \frac{-15}{l}$

Step 4
Substitute $\frac{-15}{l}$ for t into equations (1) and (2), and solve the system of equations.

① $-3 = k + 2t$
$-3 = k + 2\left(\frac{-15}{l}\right)$
$-3 = k - \frac{30}{l}$
$k = -3 + \frac{30}{l}$

② $-2k = 3 + 3t$
$-2k = 3 + 3\left(\frac{-15}{l}\right)$
$-2k = 3 - \frac{45}{l}$
$k = -\frac{3}{2} + \frac{45}{2l}$

$k = k$
$-3 + \frac{30}{l} = -\frac{3}{2} + \frac{45}{2l}$
$-6l + 60 = -3l + 45$
$-3l = -15$
$l = 5$

Step 5
Substitute the value of $l = 5$ into $k = -3 + \frac{30}{l}$ to solve for k.

① $k = -3 + \frac{30}{l}$
$k = -3 + \frac{30}{(5)}$
$k = -3 + 6$
$k = 3$

Based on the values of k and l, $k = \frac{3}{5}l$.

26. B

Step 1
Determine direction vectors.
Based on the given information, $A(p, q, r)$ or $B(2p, 3q, 4r)$ could be position vectors. Thus, the direction vectors could be either $\overrightarrow{AB}$ or $\overrightarrow{BA}$.

$\overrightarrow{AB} = (2p - p, 3q - q, 4r - r)$
$= (p, 2q, 3r)$
$\overrightarrow{BA} = (p - 2p, q - 3q, r - 4r)$
$= (-p, -2q, -3r)$

Step 2
Combine the position vectors and direction vectors to make possible parametric equations.
The following vectors and parametric equations describe the line:

$\vec{r}_1 = (p, q, r) + t(p, 2q, 3r)$
$x = p + pt$
$y = q + 2qt$
$z = r + 3rt$

$\vec{r}_2 = (p, q, r) + t(-p, -2q, -3r)$
$x = p - pt$
$y = q - 2qt$
$z = r - 3rt$

$\vec{r}_3 = (2p, 3q, 4r) + t(p, 2q, 3r)$
$x = 2p + pt$
$y = 3q + 2qt$
$z = 4r + 3rt$

$\vec{r}_4 = (2p, 3q, 4r) + t(-p, -2q, -3r)$
$x = 2p - pt$
$y = 3q - 2qt$
$z = 4r - 3rt$

Alternative B is the only one represented.

27. C

If $\vec{a}$, $\vec{b}$, and $\vec{c}$ are normal vectors of the three non-parallel planes and $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$, then the normals are coplanar and their corresponding orthogonal planes do not intersect in a point. However, if $\vec{a} \cdot (\vec{b} \times \vec{c}) \neq 0$, then the planes do intersect in a point.

Step 1
Since all four sets of planes have the same equations that define planes P_1 and P_2, find the normals $\vec{b}$ and $\vec{c}$ to these planes, and then calculate their cross product $\vec{b} \times \vec{c}$.

P_1: $2x - y + 3z - 2 = 0 \rightarrow$ Normal: $\vec{b} = (2, -1, 3)$
P_2: $x - 3y + 2z + 10 = 0 \rightarrow$ Normal: $\vec{c} = (1, -3, 2)$

Calculate the cross product $\vec{b} \times \vec{c}$.

$$\begin{aligned}\vec{b} \times \vec{c} &= (2, -1, 3) \times (1, -3, 2)\\ &= \begin{pmatrix}[(-1)(2) - (3)(-3)],\\ [(3)(1) - (2)(2)],\\ [(2)(-3) - (-1)(1)],\end{pmatrix}\\ &= [-2 + 9, 3 - 4, -6 + 1]\\ &= (7, -1, -5)\end{aligned}$$

Step 2
Find the normal, $\vec{a}$, to plane P_3 for each of the given sets of planes, and then determine the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$.

The normal, $\vec{a}$, for the set of planes A is (10, −10, 16), so the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$ is as follows:

$$\begin{aligned}&\vec{a} \cdot (\vec{b} \times \vec{c})\\ &= (10, -10, 16) \cdot (7, -1, -5)\\ &= [(10)(7) + (-10)(-1) + (16)(-5)]\\ &= 0\end{aligned}$$

Since the normal, $\vec{a}$, for the set of planes Bis the same, so is the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$.

$$\begin{aligned}&\vec{a} \cdot (\vec{b} \times \vec{c})\\ &= (10, -10, 16) \cdot (7, -1, -5)\\ &= [(10)(7) + (-10)(-1) + (16)(-5)]\\ &= 0\end{aligned}$$

The normal, $\vec{a}$, for the set of planes C is (3, 1, −1), so the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$ in this case is as follows:

$$\begin{aligned}&\vec{a} \cdot (\vec{b} \times \vec{c})\\ &= (3, 1, -1) \cdot (7, -1, -5)\\ &= [(3)(7) + (1)(-1) + (-1)(-5)]\\ &= 25\end{aligned}$$

The normal, $\vec{a}$, for the set of planes D is (3, 6, 3), so the value of $\vec{a} \cdot (\vec{b} \times \vec{c})$ in this case is as follows:

$$\begin{aligned}&\vec{a} \cdot (\vec{b} \times \vec{c})\\ &= (3, 6, 3) \cdot (7, -1, -5)\\ &= [(3)(7) + (6)(-1) + (3)(-5)]\\ &= 0\end{aligned}$$

Since $\vec{a} \cdot (\vec{b} \times \vec{c}) \neq 0$ for the normals of the non-parallel set of planes in the set of planes C, these three planes intersect at a single point.

28. B

Step 1
Determine the scalar equation of the plane.
The components of the normal vector $\vec{n} = (A, B, C)$ are the three coefficients of the scalar equation of the corresponding plane $Ax + By + Cz + D = 0$.
Therefore, since $\vec{n} = (2, -3, 1)$, the scalar equation of the plane is $2x - 3y + 1z + D = 0$.
Since the point $P(3, 0, 5)$ lies on the plane, substitute its coordinates for x, y, and z, and solve for D.

$$\begin{aligned}2x - 3y + z + D &= 0\\ 2(3) - 3(0) + 5 + D &= 0\\ 6 + 5 + D &= 0\\ D &= -11\end{aligned}$$

Thus, the scalar equation of the plane is $2x - 3y + z - 11 = 0$.

Step 2
Determine which other point lies on the plane by substituting their coordinates for x, y, and z.
Substitute (4, −7, −3) into the scalar equation of the plane.

$$\begin{aligned}2x - 3y + z - 11 &= 0\\ 2(4) - 3(-7) + (-3) - 11 &= 0\\ 8 + 21 - 3 - 11 &= 0\\ 15 &= 0\end{aligned}$$

Substitute (6, 0, −1).

$$\begin{aligned}2x - 3y + z - 11 &= 0\\ 2(6) - 3(0) + (-1) - 11 &= 0\\ 12 - 1 - 11 &= 0\\ 0 &= 0\end{aligned}$$

Substitute (2, 3, 2).

$$\begin{aligned}2x - 3y + z - 11 &= 0\\ 2(2) - 3(3) + 2 - 11 &= 0\\ 4 - 9 + 2 - 11 &= 0\\ -14 &= 0\end{aligned}$$

Substitute (4, 1, 0).

$$\begin{aligned}2x - 3y + z - 11 &= 0\\ 2(4) - 3(1) + 0 - 11 &= 0\\ 8 - 3 - 11 &= 0\\ -6 &= 0\end{aligned}$$

Thus, the point that would also lie on the plane defined by $2x - 3y + z - 11 = 0$ is the one with coordinates (6, 0, −1).

29. D

Step 1
Find the normal vectors, $\vec{n}$, to each plane.

$\vec{n}_1 = (-1, 3, 1)$

$\vec{n}_2 = (1, 1, 2)$

$\vec{n}_3 = (2, -2, 1)$

None of the normals are scalar multiples of one another, which supports the fact that they are not parallel. Therefore, the planes are distinct, or non-parallel.

Step 2
Determine whether the normal vectors are coplanar.

$$\begin{aligned}0 &= \vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3)\\ 0 &= (-1, 3, 1) \cdot [(1, 1, 2) \times (2, -2, 1)]\\ 0 &= (-1, 3, 1) \cdot (5, 3, -4)\\ 0 &= (-1)(5) + (3)(3) + (1)(-4)\\ 0 &= -5 + 9 - 4\\ 0 &= 0\end{aligned}$$

Step 3

Since $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$, the normals are coplanar and the planes intersect at a line or not at all.

Solve the system algebraically to determine whether there is a solution.

To eliminate the variable x, add equations (1) and (2) first.

$$\begin{array}{ll} ① & -x + 3y + z + 10 = 0 \\ ② & \underline{x + y + 2z - 8 = 0} \\ ④ & 4y + 3z + 2 = 0 \end{array}$$

Then, add equation (3) and 2 times equation (1).

$$\begin{array}{ll} 2 \times ① & -2x + 6y + 2z + 20 = 0 \\ ③ & \underline{2x - 2y + z - 4 = 0} \\ ⑤ & 4y + 3z + 16 = 0 \end{array}$$

Subtract equation (5) from equation (4).

$$\begin{array}{ll} ④ & 4y + 3z + 2 = 0 \\ ⑤ & \underline{4y + 3z + 16 = 0} \\ & 0y + 0z - 14 = 0 \\ & 0y + 0z = 14 \end{array}$$

This system has no solution. Therefore, the three planes are non-parallel and intersect each other in pairs (three parallel lines).

30. WR

Step 1

Find the normals, $\vec{n}$, to each plane.

$\vec{n}_1 = (1, 2, 3)$

$\vec{n}_2 = (1, -1, -3)$

$\vec{n}_3 = (1, 5, 9)$

None of the normals are scalar multiples of one another, which means that none of the planes are parallel. Therefore, the planes are distinct and intersect either in a line or at a point.

Step 2

Determine whether the normal vectors are coplanar, $\vec{n} \cdot (\vec{n}_2 \times \vec{n}_3) = 0$.

$$\begin{aligned} &\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) \\ &= (1, 2, 3) \cdot [(1, -1, -3) \times (1, 5, 9)] \\ &= (1, 2, 3) \cdot \begin{pmatrix} [(-1)(9) - (-3)(5)], \\ [(-3)(1) - (1)(9)], \\ [(1)(5) - (-1)(1)] \end{pmatrix} \\ &= (1, 2, 3) \cdot (6, -12, 6) \\ &= (1)(6) + 2(-12) + 3(6) \\ &= 6 - 24 + 18 \\ &= 0 \end{aligned}$$

Since $\vec{n}_1 \cdot (\vec{n}_2 \times \vec{n}_3) = 0$, the normal vectors are coplanar, which means that the planes intersect in a line.

31. D

Step 1

Determine the normal vector to the plane.

If the plane is perpendicular to the line defined by $(x, y, z) = (-2, 3, 0) + s(2, 3, -1)$, the direction vector of the line is a normal vector to the plane, which makes $\vec{n} = (2, 3, -1)$. Thus, the scalar equation of the plane is $2x + 3y - z + D = 0$.

Step 2

A point on the plane is $(1, 2, -3)$, since the line defined by $(1, 2, -3) + t(4, 2, -6)$ is contained in the plane. Substitute the point $(1, 2, -3)$ into the equation $2x + 3y - z + D = 0$, and find the value of D.

$$\begin{aligned} 2x + 3y - z + D &= 0 \\ 2(1) + 3(2) - (-3) + D &= 0 \\ 11 + D &= 0 \\ D &= -11 \end{aligned}$$

Therefore, the scalar equation of the plane is $2x + 3y - z - 11 = 0$.

32. C

Step 1

Determine the parametric equation of the plane.

① $x = -5 + 4t - 3s$

② $y = 2 + 8t - 2s$

③ $z = -4 + t$

Statement I is correct because it represents the position vector defined by $A(-5, 2, -4)$ and the two direction vectors $\overrightarrow{AB} = (4, 8, 1)$ and $\overrightarrow{AC} = (-3, -2, 0)$.

Step 2

Determine the vector equation of the plane.

A vector equation that describes the plane is represented by the position vector defined by the point $A(-5, 2, -4)$ and scalar multiples of the direction vectors $\overrightarrow{AB} = (4, 8, 1)$ and $\overrightarrow{AC} = (-3, -2, 0)$. Since the vector $(3, 2, 0)$ is a multiple of vector $\overrightarrow{AC}$ by a scalar value of -1, statement II is also correct. The vector equation of the plane can be defined as $(x, y, z) = (-5, 2, -4) + t(3, 2, 0) + s(4, 8, 1)$.

Step 3
Determine whether the point $D(31, 6, 0)$ lies on the plane.
Substitute 0 for z, and solve for t in equation (3).

$$
\begin{aligned}
③\quad z &= -4+t\\
0 &= -4+t\\
t &= 4
\end{aligned}
$$

Substitute 31 for x and 4 for t, and solve for s in equation (1).

$$
\begin{aligned}
①\quad x &= -5+4t-3s\\
31 &= -5+4(4)-3s\\
31 &= 11-3s\\
20 &= -3s\\
s &= -\frac{20}{3}
\end{aligned}
$$

Substitute 6 for y and 4 for t, and solve for s in equation (2).

$$
\begin{aligned}
②\quad y &= 2+8t-2s\\
6 &= 2+8(4)-2s\\
6 &= 34-2s\\
-28 &= -2s\\
s &= 14
\end{aligned}
$$

Since the values of s in equations (1) and (2) are not the same, the point (31, 6, 0) does not lie on the plane. Therefore, statement III is false.

Step 4
Determine the z-intercept.
To determine whether the z-intercept is (0, 0, −5), substitute −5 for z in equation (3).

$$
\begin{aligned}
③\quad z &= -4+t\\
-5 &= -4+t\\
t &= -1
\end{aligned}
$$

Substitute 0 for x and −1 for t, and solve for s in equation (1).

$$
\begin{aligned}
①\quad x &= -5+4t-3s\\
0 &= -5+4(-1)-3s\\
0 &= -9-3s\\
s &= -3
\end{aligned}
$$

Substitute 0 for y and −1 for t, and solve for s in equation (2).

$$
\begin{aligned}
②\quad y &= 2+8t-2s\\
0 &= 2+8(-1)-2s\\
0 &= -6-2s\\
s &= -3
\end{aligned}
$$

Since $s = -3$ and $t = -1$ for all three parts of the parametric equation, statement IV correctly defines the z-intercept as (0, 0, −5).

33. A

Method 1
Step 1
Rewrite the vector form of the equation of the plane in parametric form.

$$
\begin{aligned}
x &= 0+t+0\\
y &= -3+4t+3s\\
z &= 0+kt+2s
\end{aligned}
$$

Step 2
Substitute the expressions for x, y, and z into the scalar equation, and solve for A.

$$
\begin{aligned}
0 &= Ax-2y+3z-6\\
0 &= \begin{pmatrix} A(0+t+0)-2(-3+4t+3s)\\ +3(0+kt+2s)-6 \end{pmatrix}\\
0 &= At+6-8t-6s+3kt+6s-6\\
0 &= At-8t+3kt\\
0 &= t(A-8+3k)\\
0 &= A-8+3k\\
A &= 8-3k
\end{aligned}
$$

Method 2
Step 1
Use the two direction vectors, $(1, 4, k)$ and $(0, 3, 2)$, given in the vector form to find a normal vector, $\vec{n}$.

$$
\begin{aligned}
\vec{n} &= (1, 4, k)\times(0, 3, 2)\\
&= \begin{pmatrix} [(4)(2)-(k)(3)],\\ [(k)(0)-(1)(2)],\\ (1)(3)-(4)(0) \end{pmatrix}\\
&= (8-3k, -2, 3)
\end{aligned}
$$

Step 2
Determine the value of A.
Since the coordinates of the normal vector, $\vec{n}$, correspond to the values of A, B, and C in the scalar form $Ax + By + Cz + D = 0$ of the equation of the plane, the scalar equation would be $(8-3k)x - 2y + 3z + D = 0$.
Therefore, $A = 8 - 3k$.

34. D

Step 1
Determine the normal vector to the plane.
A point on the plane is (3, 1, −2), and two direction vectors are $\vec{u} = (5, 0, -1)$ and $\vec{v} = (0, 5, 2)$.
A vector normal to the plane, $\vec{n}$, can be determined by taking the cross product of vector $\vec{u}$ and vector $\vec{v}$.

$$
\begin{aligned}
\vec{n} &= \vec{u}\times\vec{v}\\
&= (5, 0, -1)\times(0, 5, 2)\\
&= \begin{pmatrix} [(0)(2)-(-1)(5)],\\ [(-1)(0)-(5)(2)],\\ [(5)(5)-(0)(0)] \end{pmatrix}\\
&= (5, -10, 25)
\end{aligned}
$$

Step 2
Determine the scalar equation of the plane.
Using the normal vector, (5, −10, 25), the equation in scalar form is $5x - 10y + 25z + D = 0$.
Substitute the coordinates of the point (3, 1, −2) into the equation to find the value of D.

$$\begin{aligned} 5x - 10y + 25z + D &= 0 \\ 5(3) - 10(1) + 25(-2) + D &= 0 \\ 15 - 10 - 50 + D &= 0 \\ -45 + D &= 0 \\ D &= 45 \end{aligned}$$

Therefore, the equation of the plane in scalar form is $5x - 10y + 25z + 45 = 0$.

35. C

Step 1
Based on the vector form of the equation of the plane, two direction vectors are $\vec{u} = (1, -3, 1)$ and $\vec{v} = (2, 4, -3)$. A vector normal to the plane, $\vec{n}$, can then be determined as follows:

$$\begin{aligned} \vec{n} &= \vec{u} \times \vec{v} \\ &= (1, -3, 1) \times (2, 4, -3) \\ &= [(-3)(-3) - (1)(4), (1)(2) \\ &\quad - (1)(-3), (1)(4) - (-3)(2)] \\ &= (5, 5, 10) \end{aligned}$$

Step 2
A direction vector for line L_1 is $\vec{w} = (-1, -1, 1)$.
Determine the dot product between the normal, $\vec{n}$, and the direction vector, $\vec{w}$.

$$\begin{aligned} \vec{n} \cdot \vec{w} &= (5, 5, 10) \cdot (-1, -1, 1) \\ &= (5)(-1) + (5)(-1) + (10)(1) \\ &= 0 \end{aligned}$$

Since $\vec{n} \cdot \vec{w} = 0$, $\vec{w}$ is perpendicular to $\vec{n}$, which means that the line L_1 and the plane P_1 are parallel.
This could mean that the line lies on the plane or is parallel to the plane. One way to find out is to write the equation of the plane in scalar form and then use a test point that lies on the line to see if the line also lies on the plane.

Step 3
Since $\vec{n} = (5, 5, 10)$ and a point on the plane is (4, −3, −1), you can determine the scalar equation that represents the plane, P_1.

$$\begin{aligned} 5x + 5y + 10z + D &= 0 \\ 5(4) + 5(-3) + 10(-1) + D &= 0 \\ -5 + D &= 0 \\ D &= 5 \end{aligned}$$

Therefore, the scalar equation of the plane is P_1: $5x + 5y + 10z + 5 = 0$.

Step 4
A point on the line L_1 is (3, 1, −2). Check if it lies on the plane.

$$\begin{aligned} 5(3) + 5(1) + 10(-2) + 5 &= 0 \\ 15 + 5 - 20 + 5 &= 0 \\ 5 &\neq 0 \end{aligned}$$

The point (3, 1, −2) does not lie on the plane. Therefore, the line and the plane are parallel and distinct, which means that there are no solutions.

36. a) WR

Step 1
Equate the x-, y-, and z-expressions of both lines.

$$\begin{aligned} x_1 &= x_2 \\ 2s &= 1 - 3t \\ ① \; 2s + 3t &= 1 \\ y_1 &= y_2 \\ 1 + s &= -2 - t \\ ② \; s + t &= -3 \\ z_1 &= z_2 \\ 2 - 3s &= 1 + t \\ ③ \; -3s - t &= -1 \end{aligned}$$

Step 2
Using equations (2) and (3), solve the system for s and t.
Add equation (3) to equation (2).

$$\begin{aligned} ② \quad s + t &= -3 \\ ③ \quad \underline{-3s - t} &\underline{= -1} \\ -2s &= -4 \\ s &= 2 \end{aligned}$$

Substitute 2 for s into equation (2).

$$\begin{aligned} ② \; s + t &= -3 \\ 2 + t &= -3 \\ t &= -5 \end{aligned}$$

Step 3
Substitute 2 for s and −5 for t into equation (1).

$$\begin{aligned} ① \; 2s + 3t &= 1 \\ 2(2) + 3(-5) &= 1 \\ -11 &\neq 1 \end{aligned}$$

Since there is no solution for s and t, these lines do not intersect, and therefore must be skew lines.

b) WR

Step 1
Use points P and Q on L_1 and L_2 to find a position vector $\overrightarrow{PQ}$.
A point on line L_1 is $P(0, 1, 2)$. A point on line L_2 is $Q(1, -2, 1)$.

$$\begin{aligned} \overrightarrow{PQ} &= (1, -2, 1) - (0, 1, 2) \\ &= (1, -3, -1) \end{aligned}$$

Step 2
Use the direction vectors $\vec{u}$ and $\vec{v}$ for both lines to find the normal vector, $\vec{n}$, to the lines.
The direction vector for line L_1 is $\vec{u} = (2, 1, -3)$.
For line L_2, it is $\vec{v} = (-3, -1, 1)$.

$$\begin{aligned}\vec{n} &= \vec{u} \times \vec{v} \\ &= (2, 1, -3) \times (-3, -1, 1) \\ &= \begin{pmatrix} [(1)(1) - (-3)(-1)], \\ [(-3)(-3) - (2)(1)], \\ [(2)(-1) - (1)(-3)] \end{pmatrix} \\ &= (-2, 7, 1)\end{aligned}$$

Step 3
Find the distance, d, between the two lines by carrying out a projection of $\overrightarrow{PQ}$ onto $\vec{n}$.

$$\begin{aligned}d &= \frac{|\overrightarrow{PQ} \cdot \vec{n}|}{|\vec{n}|} \\ &= \frac{|(1, -3, -1) \cdot (-2, 7, 1)|}{|-2, 7, 1|} \\ &= \frac{|-2 - 21 - 1|}{\sqrt{(-2)^2 + 7^2 + 1^2}} \\ &= \frac{24}{\sqrt{54}} \\ &= 3.265\,986\ldots\end{aligned}$$

To the nearest tenth unit, the distance between the two lines is 3.3.

37. A

Skew lines are two lines that do not intersect but are not parallel. They could intersect a plane at a single point. In this case, the plane is parallel to one of the lines, and the other line intersects the plane at one point. Skew lines could intersect a plane at two points. In this case, the plane is not parallel to any of the lines, and both lines intersect the plane at two points. Skew lines could also intersect a plane at one point and one line. In this case, the plane is coincident with one of the lines, and the other line intersects at one point on the plane. However, the intersection could never be at two lines.

NOTES

KEY Strategies for Success on Tests

TEST PREPARATION AND TEST-TAKING SKILLS

THINGS TO CONSIDER WHEN TAKING A TEST

- It is normal to feel anxious before you write a test. You can manage this anxiety by
  –Thinking positive thoughts. Imagine yourself doing well on the test.
  –Making a conscious effort to relax by taking several slow, deep, controlled breaths. Concentrate on the air going in and out of your body.
- Before you begin the test, ask questions if you are unsure of anything.
- Jot down key words or phrases from any instructions your teacher gives you.
- Look over the entire test to find out the number and kinds of questions on the test.
- Read each question closely and reread if necessary.
- Pay close attention to key vocabulary words. Sometimes these are **bolded** or *italicized*, and they are usually important words in the question.
- If you are putting your answers on an answer sheet, mark your answers carefully. Always print clearly. If you wish to change an answer, erase the mark completely and then ensure your final answer is darker than the one you have erased.
- Use highlighting to note directions, key words, and vocabulary that you find confusing or that are important to answering the question.
- Double-check to make sure you have answered everything before handing in your test.

When taking tests, students often overlook the easy words. Failure to pay close attention to these words can result in an incorrect answer. One way to avoid this is to be aware of these words and to underline, circle, or highlight them while you are taking the test.

Even though some words are easy to understand, they can change the meaning of the entire question, so it is important that you pay attention to them. Here are some examples.

all	**always**	**most likely**	**probably**	**best**	**not**
difference	**usually**	**except**	**most**	**unlikely**	**likely**

Example

1. Which of the following equations is **not** correct?

A. $3 + 2 = 5$

B. $4 - 3 = 1$

C. $5 \times 4 = 15$

D. $6 \times 3 = 18$

HELPFUL STRATEGIES FOR ANSWERING MULTIPLE-CHOICE QUESTIONS

A multiple-choice question gives you some information, and then asks you to select an answer from four choices. Each question has one correct answer. The other answers are distractors, which are incorrect. Below are some strategies to help you when answering multiple-choice questions.

- Quickly skim through the entire test. Find out how many questions there are and plan your time accordingly.
- Read and reread questions carefully. Underline key words and try to think of an answer before looking at the choices.
- If there is a graphic, look at the graphic, read the question, and go back to the graphic. Then, you may want to underline the important information from the question.
- Carefully read the choices. Read the question first and then each answer that goes with it.
- When choosing an answer, try to eliminate those choices that are clearly wrong or do not make sense.
- Some questions may ask you to select the best answer. These questions will always include words like *best*, *most appropriate*, or *most likely*. All of the answers will be correct to some degree, but one of the choices will be better than the others in some way. Carefully read all four choices before choosing the answer you think is the best.
- If you do not know the answer, or if the question does not make sense to you, it is better to guess than to leave it blank.
- Do not spend too much time on any one question. Make a mark (*) beside a difficult question and come back to it later. If you are leaving a question to come back to later, make sure you also leave the space on the answer sheet, if you are using one.
- Remember to go back to the difficult questions at the end of the test; sometimes clues are given throughout the test that will provide you with answers.
- Note any negative words like *no* or *not* and be sure your choice fits the question.
- Before changing an answer, be sure you have a very good reason to do so.
- Do not look for patterns on your answer sheet, if you are using one.

HELPFUL STRATEGIES FOR ANSWERING OPEN-RESPONSE QUESTIONS

A written response requires you to respond to a question or directive such as **explain**, **predict**, **list**, **describe**, **show your work**, **solve**, or **calculate.** In preparing for open-response tasks you may wish to:

- Read and reread the question carefully.
- Recognize and pay close attention to directing words such as *explain*, *show your work*, and *describe*.
- Underline key words and phrases that indicate what is required in your answer, such as *explain*, *estimate*, *answer*, *calculate*, or *show your work*.
- Write down rough, point-form notes regarding the information you want to include in your answer.
- Think about what you want to say and organize information and ideas in a coherent and concise manner within the time limit you have for the question.
- Be sure to answer every part of the question that is asked.
- Include as much information as you can when you are asked to explain your thinking.
- Include a picture or diagram if it will help to explain your thinking.
- Try to put your final answer to a problem in a complete sentence to be sure it is reasonable.
- Reread your response to ensure you have answered the question.
- Think: Does your answer make sense?
- Listen: Does it sound right?
- Use appropriate subject vocabulary and terms in your response.

About Mathematics Tests

What You Need to Know about Mathematics Tests

To do well on a mathematics test, you need to understand and apply your knowledge of mathematical concepts. Reading skills can also make a difference in how well you perform. Reading skills can help you follow instructions and find key words, as well as read graphs, diagrams, and tables. They can also help you solve mathematics problems.

Mathematics tests usually have two types of questions: questions that ask for understanding of mathematics ideas and questions that test how well you can solve mathematics problems.

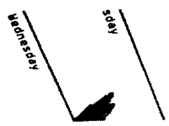
How You Can Prepare for the Mathematics Test

Below are some strategies that are particular to preparing for and writing mathematics tests.

- Know how to use your calculator and, if it is allowed, use your own for the test.
- Note-taking is a good way to review and study important information from your class notes and textbook.
- Sketch a picture of the problem, procedure, or term. Drawing is helpful for learning and remembering concepts.
- Check your answer to practice questions by working backward to the beginning. You can find the beginning by going step-by-step in reverse order.
- When answering questions with graphics (pictures, diagrams, tables, or graphs), read the test question carefully.

  –Read the title of the graphic and any key words.
  –Read the test question carefully to figure out what information you need to find in the graphic.
  –Go back to the graphic to find the information you need.
- Decide which operation is needed.
- Always pay close attention when pressing the keys on your calculator. Repeat the procedure a second time to be sure you pressed the correct keys.

  •



Test Preparation Countdown

If you develop a plan for studying and test preparation, you will perform well on tests.

Here is a general plan to follow seven days before you write a test.

Countdown: 7 Days before the Test

1. Use "Finding Out About the Test" to help you make your own personal test preparation plan.
2. Review the following information:
   - Areas to be included on the test
   - Types of test items
   - General and specific test tips
3. Start preparing for the test at least 7 days before the test. Develop your test preparation plan and set time aside to prepare and study.

Countdown: 6, 5, 4, 3, 2 Days before the Test

1. Review old homework assignments, quizzes, and tests.
2. Rework problems on quizzes and tests to make sure you still know how to solve them.
3. Correct any errors made on quizzes and tests.
4. Review key concepts, processes, formulas, and vocabulary.
5. Create practice test questions for yourself and then answer them. Work out many sample problems.

Countdown: The Night before the Test

1. The night before the test is for final preparation, which includes reviewing and gathering material needed for the test before going to bed.
2. Most important is getting a good night's rest and knowing you have done everything possible to do well on the test.

Test Day

1. Eat a healthy and nutritious breakfast.
2. Ensure you have all the necessary materials.
3. Think positive thoughts: "I can do this." "I am ready." "I know I can do well."
4. Arrive at your school early so you are not rushing, which can cause you anxiety and stress.

Summary Of How To Be Successful During A Test

You may find some of the following strategies useful for writing a test.

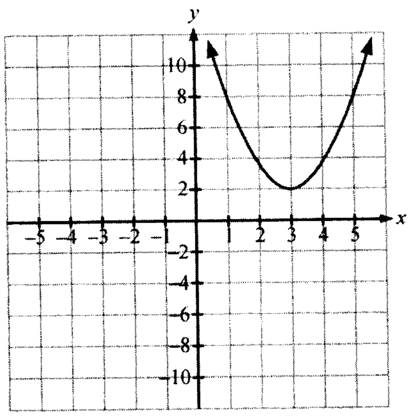
- Take two or three deep breaths to help you relax.
- Read the directions carefully and underline, circle, or highlight any important words.
- Look over the entire test to understand what you will need to do.
- Budget your time.
- Begin with an easy question, or a question you know you can answer correctly, rather than following the numerical question order of the test.
- If you cannot remember how to answer a question, try repeating the deep breathing and physical relaxation activities first. Then, move on to visualization and positive self-talk to get yourself going.
- When answering a question with graphics (pictures, diagrams, tables, or graphs), look at the question carefully.
  - Read the title of the graphic and any key words.
  - Read the test question carefully to figure out what information you need to find in the graphic.
  - Go back to the graphic to find the information you need.
- Write down anything you remember about the subject on the reverse side of your test paper. This activity sometimes helps to remind you that you do know something and you are capable of writing the test.
- Look over your test when you have finished and double-check your answers to be sure you did not forget anything.

NOTES

Practice Tests

PRACTICE TEST 1

Use the following information to answer the next question.



The function $f(x)$ is shown.

1. What is the instantaneous rate of change of the given function at the point where $x = 3$?

A. 0 **B.** $\frac{2}{3}$

C. $\frac{3}{2}$ **D.** 3

2. Which of the following pairs of x-values would be **best** to use to approximate the instantaneous rate of change of a smooth function at $x = 2$?

A. $x = 1$ and $x = 3$

B. $x = 2$ and $x = 2.5$

C. $x = 2.5$ and $x = 3$

D. $x = 1.5$ and $x = 2.5$

3. Which of the following statements about the function $f(x)$, characterized by the expression $\lim\limits_{h \to 0} \frac{f(-5+h) - f(-5)}{h} = \frac{f(4) - f(0)}{4}$, is **true**?

A. The instantaneous rate of change of $f(x)$ at $x = -5$ is equal to the instantaneous rate of change of $f(x)$ at $x = 4$.

B. The instantaneous rate of change of $f(x)$ at $x = -5$ is equal to the average rate of change of $f(x)$ between $x = 0$ and $x = 4$.

C. The average rate of change of $f(x)$ between $x = h$ and $x = -5$ is equal to the instantaneous rate of change of $f(x)$ at $x = 4$.

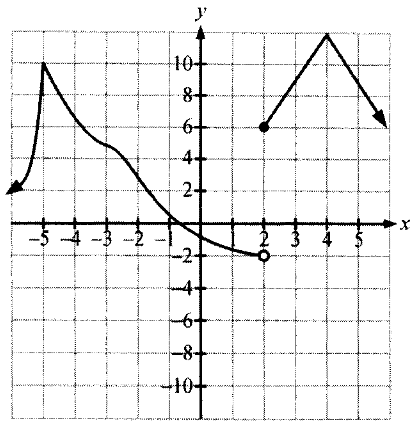
D. The average rate of change of $f(x)$ between $x = h$ and $x = -5$ is equal to the average rate of change of $f(x)$ between $x = 0$ and $x = 4$.

Numerical Response

4. The instantaneous rate of change of the function $f(x) = \frac{1}{2}x^2 - 4x$ at the point where $x = 7$ is __________.

Use the following information to answer the next question.

The graph of the function $f(x)$ is shown.

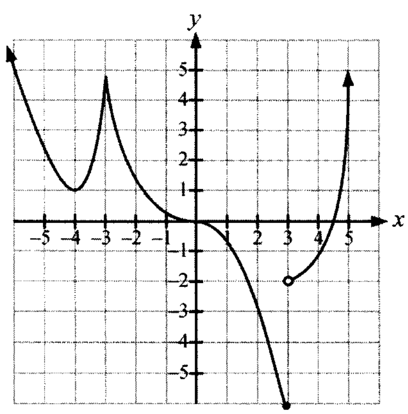


5. At which of the following x-values can a tangent line to the curve of $f(x)$ be drawn?

A. –5 **B.** –3

C. 2 **D.** 4

Use the following information to answer the next question.



6. At which x-values on the graph of f can the instantaneous rate of change **not** be determined?

A. $x = -4$ and $x = 3$

B. $x = -3$ and $x = 0$

C. $x = -4$ and $x = 0$

D. $x = -3$ and $x = 3$

Written Response

7. What function would produce the derivative given by the expression $\lim_{h \to 0} \frac{3(x+h)^2 - 8 - (3x^2 - 8)}{h}$?

Numerical Response

8. Rounded to the nearest hundredth, for what value of a in the function $f(x) = a^x$ would $f(x) = f'(x)$? _______

Written Response

9. If $\ln x = 6$, determine the exact value of x.

Written Response

10. Describe how the graph of $f'(x)$ compares to the graph of $f(x)$ for $f(x) = a^x, a > 1$.

Written Response

11. Express the ratio $\frac{f'(x)}{f(x)}$ for $f(x) = 6^x$ as an exact value.

12. Which of the following sets of equations does **not** represent a function $f(x)$, its derivative at a given value of x, and the function's power rule verification at this value of x?

A. $f(x) = x$, $f'(5) = 1$,
$$f'(5) = \lim_{h \to 0} \frac{(5+h)-5}{h}$$

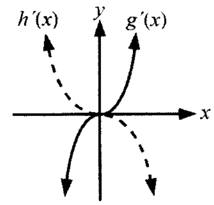
B. $f(x) = x^3$, $f'(0) = 0$,
$$f'(0) = \lim_{h \to 0} \frac{(0+h)^3 - 0^3}{h}$$

C. $f(x) = x^4$, $f'(1) = 4$,
$$f'(1) = \lim_{h \to 0} \frac{(1+h)^4 - 1^4}{h}$$

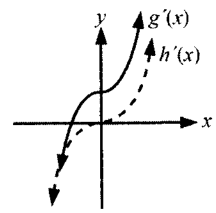
D. $f(x) = x^2$, $f'(-2) = -4$,
$$f'(-2) = \lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$$

13. If $g(x) = kh(x)$, where k is a constant and $k < 0$, which of the following graphs could be the graphs of $g'(x)$ and $h'(x)$?

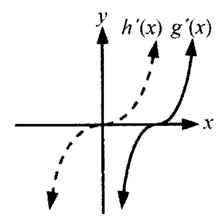
A.



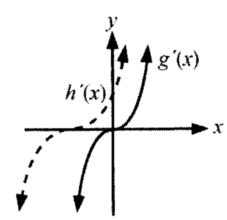
B.



C.



D.



Written Response

14. Let f be the function defined by $f(x) = 2x^3 - \frac{11}{2}x^2 + 5x$. Determine the coordinates of the points, rounded to the nearest tenth, where the slope of the tangent is equal to 15.

15. The tangent line to $f(x) = \sqrt{x}$ at the point where $x = 4$ has a slope of

A. $\frac{1}{4}$ **B.** $\frac{1}{2}$

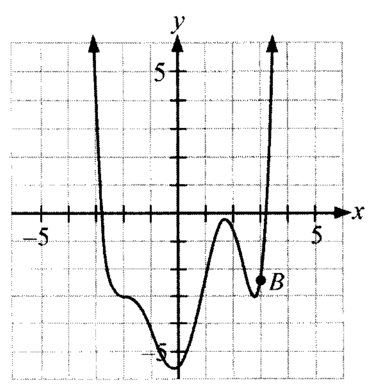
C. 1 **D.** 4

Written Response

16. Determine the derivative of the function $f(x) = e^{x^2} \cos(2x + 1)$.

Use the following information to answer the next question.

Point B is labelled on this graph of a function.



17. Respectively, the first and second derivatives at point B are

A. $f'(x) > 0$ and $f''(x) > 0$

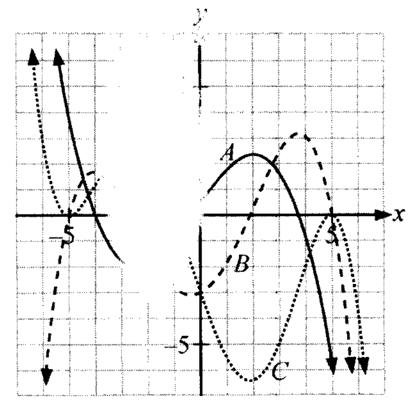
B. $f'(x) < 0$ and $f''(x) < 0$

C. $f'(x) > 0$ and $f''(x) < 0$

D. $f'(x) < 0$ and $f''(x) > 0$

Use the following information to answer the next question.

The graphs of $f(x)$, $f'(x)$, and $f''(x)$ are shown.



Written Response

18. Identify each curve as $f(x)$, $f'(x)$, or $f''(x)$. Justify your choices.

Use the following information to answer the next question.

A polynomial function is defined by $f(x) = x^3 + ax^2$.

Written Response

19. For what value of a does the graph of $f(x)$ have a point of inflection at $x = 6$?

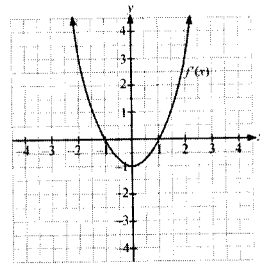
20. Which of the following x-coordinates determine the local extrema of a function with a first derivative of $f'(x) = 5x^4 + 20x^3$?

A. $x = 0, -2$ **B.** $x = 4, -1$

C. $x = 3, -1$ **D.** $x = 0, -4$

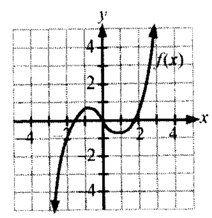
Use the following information to answer the next question.

The graph of the derivative of $f(x)$ is shown.

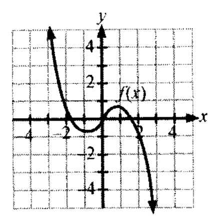


21. Which of the following graphs could be the graph of $f(x)$?

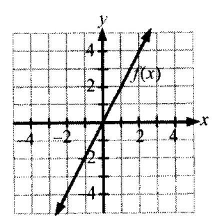
A.



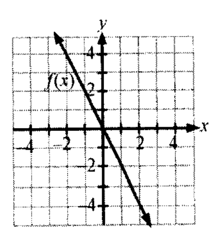
B.



C.



D.



Use the following information to answer the next question.

The first and second derivatives of a function $f(x)$ are defined by
$f'(x) = x^2(x-1)$ and
$f''(x) = x(3x-2)$.

22. The graph of $f(x)$ is both increasing and concave up over the interval

A. $-\infty < x < 0$ **B.** $0 < x < \frac{2}{3}$

C. $\frac{2}{3} < x < 1$ **D.** $1 < x < \infty$

23. A local maximum for a polynomial function $f(x)$ occurs when

A. $f'(x) = 0$ and $f''(x) = 0$

B. $f'(x) = 0$ and $f''(x) > 0$

C. $f'(x) = 0$ and $f''(x) < 0$

D. $f'(x) = 0$ and $f''(x)$ is undefined

Use the following information to answer the next question.

The second derivative of a polynomial function is zero when $x = \frac{1}{2}$.

24. If $f\left(\frac{1}{2}\right) = -\frac{1}{16}$, which of the following statements about the graph of $f(x)$ at the point $\left(\frac{1}{2}, -\frac{1}{16}\right)$ is **true**?

A. $f(x)$ is decreasing at the point $\left(\frac{1}{2}, -\frac{1}{16}\right)$.

B. $f(x)$ is concave down at the point $\left(\frac{1}{2}, -\frac{1}{16}\right)$.

C. $f(x)$ has a local minimum at the point $\left(\frac{1}{2}, -\frac{1}{16}\right)$.

D. $f(x)$ has a point of inflection at the point $\left(\frac{1}{2}, -\frac{1}{16}\right)$.

Use the following information to answer the next question.

The position function of a particle is given by $s(t) = t^3 - 4.5t^2 - 7t$, $t \geq 0$, where s is in metres and t is in seconds.

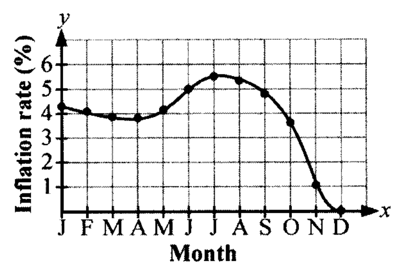
25. The particle will first reach a velocity of 5 m/s after

A. 4.0 s **B.** 5.8 s

C. 22.5 s **D.** 23.0 s

Use the following information to answer the next question.

A graph of the monthly inflation rates of prices of goods in the United States in 2008 is shown.



Written Response

26. During which 4-month period of 2008 were prices in the United States the **most stable** but still increasing?

Use the following information to answer the next question.

A train was derailed, and many cars that contained oil were damaged, causing oil to leak into a nearby lake. Environmental officials used an oil recovery pump to remove oil from the lake, and the total amount of oil (in cubic metres) that the pump removed after t hours was modelled by the function $T = \frac{250t^{\frac{3}{2}}}{3}$. By the time the pump was set up and began to work, the rate of oil leakage had reached a constant stream of 625 m^3/h.

27. How long did it take the pump to begin removing oil at a rate equal to the rate the oil was leaking?

A. 3 h **B.** 5 h

C. 25 h **D.** 26 h

Use the following information to answer the next question.

A rectangular school yard must be reseeded to repair the damage caused by a school year of activities. To protect the yard, it will be enclosed by a fence on three of its sides. The fourth side backs onto a straight ravine and is not accessible to the public.

Written Response

28. If the perimeter of the yard requiring a fence is given by the equation $P = 2x + y$, and the yard's area is given by the formula $A = xy$, in which x is the length of the yard, and y is its width, what is the maximum area that can be enclosed by 300 m of fencing?

Use the following information to answer the next question.

A toy manufacturer determined that for one month of production, its costs (C) and revenue (R) for producing x dolls of a particular type are determined by $C(x) = 38 + 12x + 0.002x^2$ and $R(x) = 20x - 0.002x^2$.

Written Response

29. How many dolls must be sold to maximize monthly profits?

Use the following information to answer the next question.

A population of rabbits grows according to the function $C(t) = Ae^{kt}$, where A is the initial number of rabbits in the population, k is a constant, and t is the time measured in years. It is also known that the population of rabbits doubles in 2.5 years.

30. What is the approximate rate of growth of the population at the instant when the population has tripled?

A. 0.83 rabbits/a

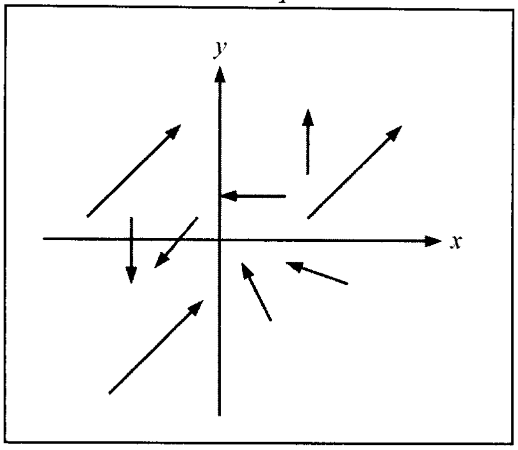
B. 3.96 rabbits/a

C. 0.83A rabbits/a

D. 3.96A rabbits/a

Practice TEST

Use the following information to answer the next question.



31. How many of the vectors shown in the given diagram appear to be equal?

A. 0 **B.** 2

C. 3 **D.** 4

32. The polar form of the Cartesian vector (8, 6) is approximately

A. (10, 36.9°)

B. (10, 41.0°)

C. (9.6, 42.5°)

D. (9.0, 38.0°)

CHALLENGER QUESTION

Numerical Response

33. If $\vec{u} = (2, -4, 6)$, $2\vec{u} + \vec{v} = (8, -3, 5)$, and $\vec{v} = (a, b, c)$, then the value of b is ______.

Use the following information to answer the next multipart question.

34. A frog hops 4 times west and then 6 times north. Each hop is approximately 1.2 m in length.

Written Response

a) To the nearest hundredth metre, how far is the frog from its original position after these 10 hops?

b) To the nearest tenth of a degree, what is the bearing of the frog's resultant displacement?

Use the following information to answer the next question.

A force of $\vec{F} = (200, 500, 200)$ in newtons acts on an object moving through a displacement of $\vec{d} = (4, -3, c)$.
The work done on the object is 7 200 J.

Numerical Response

35. To the nearest tenth, the value of c is

_________.

36. Given $\vec{a} = (4, 0, 0)$ and $\vec{b} = (4, 3, 5)$, which of the following equations **cannot** be used to find the measure of the angle, θ, between these two vectors?

A. $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

B. $\sin\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

C. $\tan\theta = \dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}|^2}$

D. $\cot\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$

37. What is the result of the cross product $(5\sqrt{2}, 2, -2\sqrt{2}) \times (5, \sqrt{2}, -2)$?

A. $(25\sqrt{2}, 2\sqrt{2}, -4\sqrt{2})$

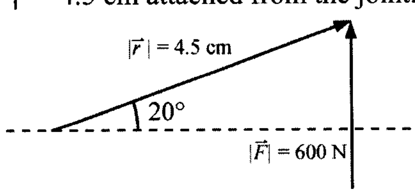
B. $(0, -20 - 10\sqrt{10}, 0)$

C. $(-8, -20\sqrt{2}, 0)$

D. $(0, 0, 0)$

Use the following information to answer the next question.

When a person lifts an object, a torque is applied by the biceps muscle on the lower arm. The elbow acts as the axis of rotation through the joint. The given diagram shows a force, $|\vec{F}|$ = 600 N, that lifts an object to the horizontal with the muscle $|\vec{r}|$ = 4.5 cm attached from the joint.



38. Which of the following equations represents the torque, $|\vec{\tau}|$, measured in $N\cdot m$?

A. $|\vec{\tau}| = (0.045)(600)\cos 70°$

B. $|\vec{\tau}| = (0.045)(600)\cos 20°$

C. $|\vec{\tau}| = (0.045)(600)\sin 70°$

D. $|\vec{\tau}| = (0.045)(600)\sin 20°$

Use the following information to answer the next question.

The equations of two lines are given.
l_1: $3x + y - 2 = 0$
l_2: $2Ax - 3By + 24 = 0$

Written Response

39. Determine the values of A and B that would cause the two lines to be coincident.

40. Which of the following points does **not** lie on the plane defined by the equation $2x - y - 8 = 0$?

A. $(2, -4, 0)$

B. $(0, -8, 8)$

C. $(4, 0, -2)$

D. $(-4, 0, 5)$

Use the following information to answer the next multipart question.

41. The parametric equation of a line is given.
$x = \sqrt{2} - \sqrt{3}t$
$y = -\sqrt{2} + 2\sqrt{3}t$

Written Response

a) Determine the x-intercept of the equation.

Practice TEST

b) Determine the scalar form of the equation.

Use the following information to answer the next question.

A teacher defined a line in three space by using the two intersecting planes shown.
P_1: $3x - 2y + z - 5 = 0$
P_2: $x - 2y - z + 2 = 0$

42. Which of the following parametric equations could also define the line?

A. $x = t$, $y = t - 3$, $z = -t + \frac{7}{2}$

B. $x = t$, $y = t + \frac{3}{4}$, $z = -t - 5$

C. $x = t + \frac{3}{4}$, $y = t$, $z = -t + \frac{11}{4}$

D. $x = \frac{7}{2} - t$, $y = -t + 11$, $z = t$

43. The scalar equation defining a plane is $\frac{3}{4}x - \frac{1}{2}y + 3z - 8 = 0$. Which of the following vectors is perpendicular to this plane?

A. $\vec{u} = \left(-\frac{2}{3}, 1, -6\right)$

B. $\vec{v} = \left(1, -\frac{2}{3}, 4\right)$

C. $\vec{w} = (4, 0, -1)$

D. $\vec{x} = (8, 2, 1)$

Use the following information to answer the next question.

A plane contains the line defined by $(x, y, z) = (1, 0, 2) + t(3, -2, 1)$ and has a y-intercept at $(0, 5, 0)$.

44. Which of the following scalar equations defines this plane?

A. $-x + 5y + 13z - 25 = 0$

B. $-x - 5y - 12z + 25 = 0$

C. $3x + 5y + 11z - 25 = 0$

D. $5x - 5y - 10z + 25 = 0$

Use the following information to answer the next question.

The scalar equation $2x + 3z - 12 = 0$ represents a plane parallel to the y-axis. The x-intercept can be represented by $\vec{u}$, and the z-intercept can be represented by $\vec{v}$.

45. Which of the following vector equations does **not** represent this plane?

A. $(x, y, z) = \vec{v} + t(\vec{u} - \vec{v}) + s(6, 0, 0)$

B. $(x, y, z) = \vec{u} + t(\vec{u} - \vec{v}) + s(0, 4, 0)$

C. $(x, y, z) = \vec{u} + t(\vec{v} - \vec{u}) + s(3, 0, 2)$

D. $(x, y, z) = \vec{v} + t(\vec{v} - \vec{u}) + s(-3, 0, 6)$

Use the following information to answer the next question.

The given equations define a line, L_1, and a plane, P_1. $L_1: (x, y, z) = (-1, 4, 2) + s(2, 1, -3)$ $P_1: -2x + y + 3x + 6 = 0$

Written Response

46. Determine the intersection point between the line and the plane.

47. A system of three planes **cannot** be

A. consistent with one solution

B. inconsistent with no solutions

C. inconsistent with two solutions

D. consistent with infinite solutions

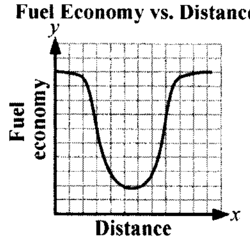
PRACTICE TEST 2

Use the following information to answer the next question.

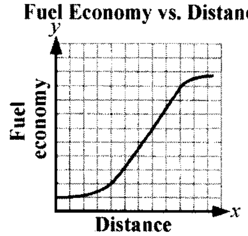
Mr. Green is driving his car at a constant speed at the bottom of a valley. Then, he drives up a hill out of the valley. Mr. Green tracks his rate of fuel consumption. He knows that his fuel consumption is the reciprocal quantity of the fuel economy.

1. Which of the following graphs represents Mr. Green's fuel economy with respect to the distance travelled?

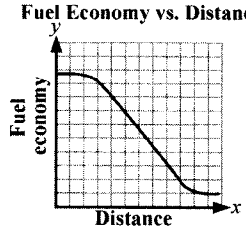
A. **Fuel Economy vs. Distance**



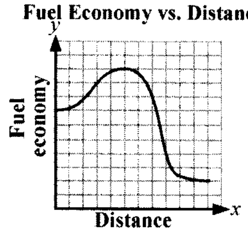
B. **Fuel Economy vs. Distance**



C. **Fuel Economy vs. Distance**



D. **Fuel Economy vs. Distance**



Use the following information to answer the next question.

A table of values for a particular function is given.

x	$f(x)$
0	4
1	5
5	25
10	50
16	110
23	200

2. Over which of the following intervals is the average rate of change of the function the **greatest**?

A. $x = 0$ to $x = 1$

B. $x = 5$ to $x = 10$

C. $x = 0$ to $x = 23$

D. $x = 16$ to $x = 23$

Numerical Response

3. To the nearest tenth, the value of $\lim_{x \to \pm\infty}\left(\frac{7x^2+4}{2x^2-8}\right)$ is ________.

4. Which of the following statements about the graph of a smooth function $f(x)$, where $\lim_{h \to 0} \frac{f(-3+h)-f(-3)}{h} = 0$, is **true**?

A. The graph of $f(x)$ is the vertical line $x = -3$.

B. The graph of $f(x)$ is the horizontal line $y = -3$.

C. The graph of $f(x)$ passes through the point $(0, -3)$.

D. The graph of $f(x)$ has a horizontal tangent at $x = -3$.

Use the following information to answer the next question.

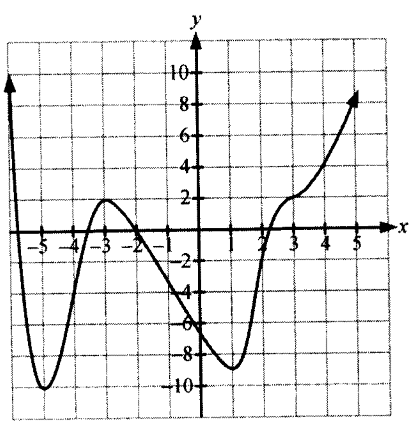
> Cassidy wanted to approximate the instantaneous rate of change of the function $f(x) = x^2 - 2x + 5$ at $x = 3$.

Numerical Response

5. To the nearest hundredth, the instantaneous rate of change of f at $x = 3$ based on $x = 3.01$ is ________.

Use the following information to answer the next question.

> The instantaneous rate of change for the function f in the given graph is ___*i*___ on $x < -5$, ___*ii*___ on $x > 3$, ___*iii*___ at $x = -3$, and ___*iv*___ at $x = 2$.



6. Which of the following tables contains the information that completes the given statement?

A.

i	*ii*	*iii*	*iv*
positive	negative	zero	negative

B.

i	*ii*	*iii*	*iv*
negative	positive	zero	positive

C.

i	*ii*	*iii*	*iv*
positive	zero	positive	negative

D.

i	*ii*	*iii*	*iv*
negative	positive	negative	zero

7. Which of the following expressions is a simplified form of

$$\lim_{h \to 0} \frac{2(x+h)^2 - 6(x+h) - (2x^2 - 6x)}{h}?$$

A. $4x - 6$

B. $-4x^2 - 12$

C. $\dfrac{2x^2 + 2h^2}{h}$

D. $\dfrac{4xh + 2h^2 - 12}{h}$

Written Response

8. For the function $f(x) = \left(\frac{1}{3}\right)^x$, describe the transformations required to transform the graph of $f(x)$ to the graph of its derivative, $f'(x)$.

Numerical Response

9. To the nearest tenth, for what value of x is the derivative of the function $f(x) = e^x$ equal to 29.8? ________

Written Response

10. Solve for x in the equation $\ln(\ln x) = 1$, and express the solution as an exact value.

Practice TEST

11. An expression that could be used to determine the instantaneous rate of change of the function $f(x)=\left(\frac{1}{2}\right)^x$ for any value of x is

A. $\dfrac{2^{-x-h}-2^{-x}}{h}$

B. $\dfrac{\left(\frac{1}{2}\right)^{x+h}-\left(\frac{1}{2}\right)^x}{h}$

C. $\lim\limits_{h\to 0}\dfrac{2^{x+h}-2x}{h}$

D. $\lim\limits_{h\to 0}\dfrac{2^{-x-h}-2^{-x}}{h}$

12. For which of the following functions does the power rule for finding the derivative **not** hold?

A. $f(x)=x$ **B.** $f(x)=2^x$

C. $f(x)=x^2$ **D.** $f(x)=x^3$

13. Which of the following expressions verifies the sum rule for the derivative of the sum of $f(x)=-2x$ and $g(x)=-5x^3$?

A. $\lim\limits_{h\to 0}\dfrac{-2(x+h)+2x}{h}+\lim\limits_{h\to 0}\dfrac{-5(x+h)^3+5x^3}{h}=\lim\limits_{h\to 0}\dfrac{\left[-2(x+h)-5(x+h)^3\right]+7x^3}{h}$

B. $\lim\limits_{h\to 0}\dfrac{-2(x+h)-2x}{h}+\lim\limits_{h\to 0}\dfrac{-5(x+h)^3-5x^3}{h}=\lim\limits_{h\to 0}\dfrac{\left[-2(x+h)-5(x+h)^3\right]-7x^3}{h}$

C. $\lim\limits_{h\to 0}\dfrac{-2(x+h)-2x}{h}+\lim\limits_{h\to 0}\dfrac{-5(x+h)^3-5x^3}{h}=\lim\limits_{h\to 0}\dfrac{\left[-2(x+h)-5(x+h)^3\right]-2x-5x^3}{h}$

D. $\lim\limits_{h\to 0}\dfrac{-2(x+h)+2x}{h}+\lim\limits_{h\to 0}\dfrac{-5(x+h)^3+5x^3}{h}=\lim\limits_{h\to 0}\dfrac{\left[-2(x+h)-5(x+h)^3\right]+2x+5x^3}{h}$

14. If $f(x)=4x^3-5x+3$, which of the following equations is the equation of the tangent line to $f(x)$ at $x=-1$?

A. $y=7x+1$

B. $y=4x+8$

C. $y=7x+11$

D. $y=-17x-13$

Written Response

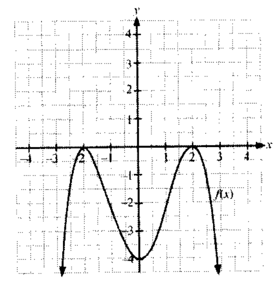
15. How can the function $f(x) = \left(-4x^{27}\right)^{\frac{1}{3}}$ be simplified to enable verification of the chain rule for finding the derivative?

Written Response

16. Determine the slope of the tangent line to the graph of $y = \dfrac{x^3}{\cos x}$ at $x = \pi$.

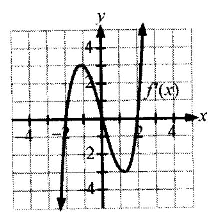
Use the following information to answer the next question.

The graph of a quartic function, $f(x)$, is shown.

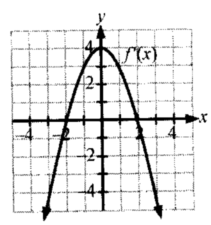


17. Which of the following graphs approximately represents the graph of the derivative of $f(x)$?

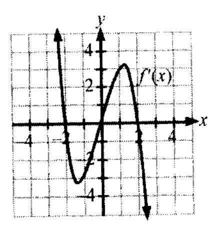
A.



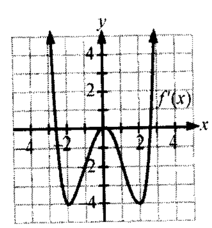
B.



C.

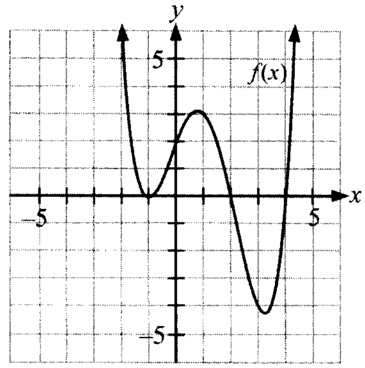


D.



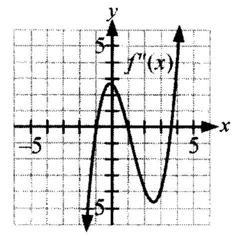
Use the following information to answer the next question.

The graph of a quartic function, $f(x)$, is shown.

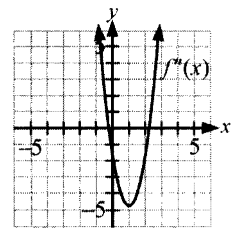


18. Which of the following graphs **best** represents the graph of $f''(x)$, the second derivative of $f(x)$?

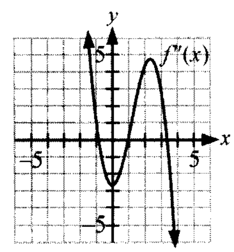
A.



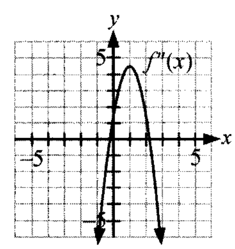
B.



C.



D.



19. The graph of a function with a first derivative of $f'(x) = 12x^5 + 15x^4$ has points of inflection for which of the following values of x?

A. $x = 0$ and $x = -\frac{3}{2}$

B. $x = 0$ and $x = -\frac{5}{4}$

C. $x = 0$ and $x = -1$

D. $x = 0$ and $x = -\frac{4}{5}$

Written Response

20. What information about a polynomial function $f(x)$ would be required to determine whether its graph lies above or below its tangent line at some point $x = a$?

Written Response

21. A certain function is defined by $f(x) = x^{101} + x^{51} + x - 1$. Show that the graph of $f(x)$ has exactly one x-intercept.

Use the following information to answer the next question.

The sign of the first derivative of a polynomial function is shown in the given table.

Interval	$f'(x)$
$-\infty < x < -1$	+
$-1 < x < 0$	−
$0 < x < 1$	+
$1 < x < \infty$	−

22. Which of the following statements about the graph of the polynomial function is **true**?

A. It is increasing on $-1 < x < 0$ and $1 < x < \infty$, and it has a local minimum at $x = 0$.

B. It is decreasing on $-1 < x < 0$ and $1 < x < \infty$, and it has a local minimum at $x = 1$.

C. It is decreasing on $-\infty < x < -1$ and $0 < x < 1$, and it has a local maximum at $x = 0$.

D. It is increasing on $-\infty < x < -1$ and $0 < x < 1$, and it has a local maximum at $x = -1$.

23. Which of the following functions is odd?

A. $f(x) = x^2 - 3x + 2$

B. $f(x) = -x^4 - x^3 - x^2$

C. $f(x) = 5x^5 + 2x^3 - 3x$

D. $f(x) = 3x^3 + 4x^2 - 5x - 1$

24. If v represents velocity and t represents time, the expression $\frac{dv}{dt}$ could be used to represent the

A. velocity function

B. position function

C. acceleration function

D. average velocity function

Use the following information to answer the next question.

At a resort in Dubai, the tide takes sand from the beach at a rate defined by $R(t) = 4 + 9\sin\left(\frac{5\pi t}{31}\right)$. Consequently, sand must be continuously added to the beach between midnight and 6 A.M.
A special underwater pump adds sand to the beach at a rate defined by $A(t) = \frac{30t}{1+2t}$, where, in both functions, $R(t)$ and $A(t)$ are in cubic metres of sand per hour and t is measured in hours after midnight.

Numerical Response

25. To the nearest hundredth, the rate at which the total amount of sand on the beach is changing at 4 A.M. is ________ m^3/h.

Use the following information to answer the next question.

The depth of water in a particular tropical bay varies throughout the day with the tides. The depth, D, of the water, in metres, can be approximated by the function $D = 2.1\cos(0.52(t - 5)) + 6.8$, in which t is the time of day according to the 24-hour clock.

26. At what time of the morning, to the nearest hour, is the rate of change of water depth decreasing the fastest?

A. 2 A.M. **B.** 5 A.M.

C. 8 A.M. **D.** 11 A.M.

Use the following information to answer the next question.

Hospital patients experiencing great pain are often prescribed morphine. When morphine is absorbed into the bloodstream, the reaction by the body to a single dose is represented by the function $B = M^2\left(k - \frac{M}{2}\right)$, in which B is the patient's blood pressure, M is the amount of morphine absorbed into the bloodstream in millilitres, and k is a constant.

27. A body's sensitivity to morphine is defined as $\frac{dB}{dM}$. What amount of morphine will cause the maximum blood pressure?

A. $\frac{3}{4}k$ mL **B.** $\frac{4}{3}k$ mL

C. $\frac{3}{2}k$ mL **D.** $4k$ mL

Practice TEST

Use the following information to answer the next question.

A race requires participants to first swim from a platform on a lake to the shore. The platform is located 4 km from the nearest point on shore. Next, participants must run from where they exited the water to the finish line, which lies along the shoreline 8 km from the point on the shore directly across from the platform.

Written Response

28. To minimize total racing time, at what point along the shore should a particular participant exit the water and begin running if she can swim at a rate of 3 km/h and can run 5 km/h?

29. Which of the following actions represents a vector quantity?

A. A woman's mass is measured to be 60 kg.

B. A car with a speed of 2.00 m/s.

C. A balloon rises vertically at 1.50 km/h.

D. A liquid with a density of 1.0 g/ml

30. What is the Cartesian form of a vector that represents a force of 233.0 N at a bearing of 72.0°?

A. (233.0, 72.0) **B.** (221.6, 72.0)

C. (72.0, 233.0) **D.** (72.0, 221.6)

Numerical Response

31. If $\vec{p} = (-3, a)$, $\vec{q} = (4, 6)$, and $2\vec{p} + 3\vec{q} = (6, 28)$, then the value of a is __________.

Use the following information to answer the next question.

Team A and team B are having a tug of war. Four members of team A pull with forces of 215 N, 230 N, 250 N, and 280 N east. Three of the members of team B each pull with a force of 235 N west. In order for team B to win, the net force must be 105 N west.

32. With what force does the fourth member of team B need to pull in order for them to win?

A. 105 N west **B.** 165 N west

C. 270 N west **D.** 375 N west

Numerical Response

33. To the nearest tenth, the magnitude of the projection of $\vec{u} = (12, 5)$ on $\vec{v} = (3, 4)$ is __________.

Use the following information to answer the next question.

Three forces act on an object such that they are all perpendicular to one another. The forces are given as $\vec{F}_1 = (-5, 20, 25)$, $\vec{F}_2 = (20, 10, -4)$, and $\vec{F}_3 = (a, b, 45)$.

Written Response

34. Determine the values of a and b for the third force $\vec{F}_3 = (a, b, 45)$.

Practice TEST

Use the following information to answer the next question.

Vectors $\vec{u} = (a, b, c)$ and $\vec{v} = (d, e, f)$ are non-zero and not collinear.

35. An orthogonal vector to both $\vec{u}$ and $\vec{v}$ can be determined by calculating

A. $\vec{u} \cdot \vec{v}$

B. $\vec{u} \times \vec{v}$

C. $\dfrac{\vec{u} \cdot \vec{v}}{|\vec{u} \times \vec{v}|}$

D. $\dfrac{|\vec{u} \times \vec{v}|}{\vec{u} \cdot \vec{v}}$

Use the following information to answer the next multipart question.

36.

A parallelepiped is defined by $\vec{u} = (2, 0, -6)$, $\vec{v} = (0, 3, -2)$, and $\vec{w} = (9, -2, 0)$, where the base is formed by $\vec{u}$ and $\vec{v}$.

Written Response

a) Is vector $\vec{w}$ orthogonal to the base of the parallelepiped?

b) Determine the volume of the parallelepiped.

Use the following information to answer the next question.

A linear system of equations is shown. $(x, y) = (1, 4) + k(2, -3)$ $(x, y) = (3, 1) + l(9, -3)$

Written Response

37. Use an algebraic method to determine the number of solutions to the given system of equations.

38. The geometric relationship between the two planes defined by $x + y - 4 = 0$ and $x + z - 4 = 0$ is that they

A. are parallel

B. are coincident

C. intersect at a line

D. intersect at a point

Use the following information to answer the next question.

The vector form of a line in two space is given as $\vec{r} = (-8, 4) + t(2, -3)$.

39. Which of the following equations is the scalar form of this line?

A. $2x + 3y + 4 = 0$

B. $-2x + 3y - 8 = 0$

C. $3x + 2y + 16 = 0$

D. $-3x + 2y + 32 = 0$

40. The x-intercept of the line passing through the points $P(3, -2, -6)$ and $Q(2, -1, -3)$ is

A. $(1, 0, 0)$

B. $(2, 0, 0)$

C. $(5, 0, 0)$

D. $(13, 0, 0)$

Use the following information to answer the next question.

Three planes are given.
P_1: $6x + 4y - 2z + 10 = 0$
P_2: $15x + 10y - 5z + 25 = 0$
P_3: $-3x - 2y + z - 10 = 0$

Written Response

41. Determine the configuration of the three planes.

42. Which of the following equations is the scalar equation of a plane with a normal vector $\vec{n} = (-3, 0, 5)$ and a point on the plane $P(6, 3, 1)$?

A. $-3x + 5z - 13 = 0$

B. $-3x + 5z + 13 = 0$

C. $6x + 3y + z - 13 = 0$

D. $6x + 3y + z + 13 = 0$

43. What is the parametric form of the plane defined by $3x - 4y + 2z - 12 = 0$?

A. $x = 30t$, $y = -3s + 18t$, $z = 6 - 6s - 9t$

B. $x = 30t$, $y = 3s - 18t$, $z = 6 + 6s + 9t$

C. $x = 30t$, $y = -18s + 3t$, $z = 6 - 9s + 9t$

D. $x = -30t$, $y = -18s + 3t$, $z = 6 - 9s + 6t$

Use the following information to answer the next question.

The given equations represent a plane, P_1, and a line, L_1.
P_1: $(x, y, z) = (2, 1, -2) + s(2, 1, 3) + t(2, -1, -1)$
L_1: $(x, y, z) = (3, 4, 1) + k(-4, 2, 1)$

Written Response

44. Determine the geometric relationship between the plane and the line.

Written Response

45. The second derivative, $f''(x)$, of a function, $f(x)$, is found to be negative on some interval $a < x < b$. What conclusions can be drawn from this information about the graph of the function $f(x)$ on $a < x < b$?

Use the following information to answer the next question.

The surface area of a rectangular open-topped box with square base is given by $SA = x^2 + 4xy$, where x is the length of the base and y is the height of the box. An open-topped rectangular display box with a square base is to be constructed from 27 m^2 of aluminum.

Written Response

46. If volume is given by $V = x^2y$, what dimensions should be used to yield a box with maximum volume?

Practice TEST

ANSWERS AND SOLUTIONS — PRACTICE TEST 1

1. A	**11. WR**	**21. A**	**31. C**	**40. D**
2. D	**12. D**	**22. D**	**32. A**	**41. a) WR**
3. B	**13. A**	**23. C**	**33. 5**	**b) WR**
4. 3	**14. WR**	**24. D**	**34. a) WR**	**42. C**
5. B	**15. A**	**25. A**	**b) WR**	**43. B**
6. D	**16. WR**	**26. WR**	**35. 39.5**	**44. A**
7. WR	**17. A**	**27. C**	**36. D**	**45. B**
8. 2.72	**18. WR**	**28. WR**	**37. D**	**46. WR**
9. WR	**19. WR**	**29. WR**	**38. C**	**47. C**
10. WR	**20. D**	**30. C**	**39. WR**	

1. A

If a tangent line were drawn through the point where $x = 3$, it would be a horizontal line with a slope of 0. Therefore, the instantaneous rate of change at $x = 3$ is 0.

2. D

To best approximate the instantaneous rate of change of the smooth function at $x = 2$, you choose points closer and closer to $x = 2$ on both sides of the graph. The closest pair of points (excluding the point at $x = 2$) on both sides is at $x = 1.5$ and $x = 2.5$.

3. B

The expression $\lim\limits_{h \to 0} \frac{f(a+h) - f(a)}{h}$ describes the instantaneous rate of change at $x = a$, and the expression $\frac{f(a+h) - f(a)}{h}$ describes the average rate of change over the interval $a \le x \le a + h$. Therefore, the instantaneous rate of change of $f(x)$ at $x = -5$ is equal to the average rate of change of $f(x)$ between $x = 0$ and $x = 4$.

4. 3

The instantaneous rate of change at $x = a$ for a function $f(x)$ can be determined using the expression $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Determine the instantaneous rate of change at $x = 7$ for $f(x) = \frac{1}{2}x^2 - 4x$.

$$\lim_{h \to 0} \frac{f(7+h) - f(7)}{h}$$
$$= \lim_{h \to 0} \frac{\left(\frac{1}{2}(7+h)^2 - 4(7+h)\right) - \left(\frac{1}{2}(7)^2 - 4(7)\right)}{h}$$
$$= \lim_{h \to 0} \frac{\left(\frac{1}{2}(49 + 14h + h^2) - 28 - 4h\right) - \left(\frac{49}{2} - 28\right)}{h}$$
$$= \lim_{h \to 0} \frac{\left(\frac{49}{2} + 7h + \frac{h^2}{2} - 28 - 4h\right) - \left(\frac{49}{2} - 28\right)}{h}$$
$$= \lim_{h \to 0} \frac{3h + \frac{h^2}{2}}{h}$$
$$= \lim_{h \to 0} \frac{h\left(3 + \frac{h}{2}\right)}{h}$$
$$= 3$$

5. B

A tangent line can be drawn at $x = -3$, since the graph is smooth and continuous at that point.

Tangent lines cannot be drawn where corners or discontinuities exist on the graph of the function, so they cannot be drawn at $x = -5$, 2, or 4.

6. D

Since the instantaneous rate of change (slope) at $x = -3$ is both positive and negative, the rate of change cannot be found at this point. Similarly, since the slope is negative and non-existent at $x = 3$, the rate of change cannot be found at this point either.

7. WR

The derivative of a function, $f(x)$, is as follows:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

In the expression for the derivative of $f(x)$, the following is true:

$$f'(x) = \lim_{h \to 0} \frac{3(x+h)^2 - 8 - (3x^2 - 8)}{h}$$

Therefore, the function according to this expression is $f(x) = 3x^2 - 8$.

8. 2.72

The derivative of $f(x) = e^x$ is $f'(x) = e^x$. It is the only function whose derivative is the function itself.

For the function $f(x) = a^x$ to be equal to its derivative, a must equal e. The actual value of e (Euler's number) is an irrational number where the first 13 decimal places are 2.718 281 8284 (59).

To the nearest hundredth, the value of e is 2.72.

9. WR

Convert $\ln x = 6$ into an exponential equation.

$$\begin{aligned} \ln x &= 6 \\ e^{\ln x} &= e^6 \\ x &= e^6 \end{aligned}$$

10. WR

When $a > 1$, then the derivative of $f(x) = a^x$ is $f'(x) = a^x \ln a$. Thus, the graph of $f'(x)$ is the graph of $f(x) = a^x$ stretched vertically with respect to the x-axis by a factor of $\ln a$.

11. WR

The derivative $f(x) = 6^x$ is $f'(x) = 6^x \ln(6)$.
Find the quotient.

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{6^x \ln 6}{6^x} \\ &= \ln 6 \end{aligned}$$

12. D

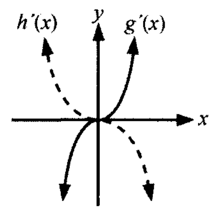
To verify the power rule, the definition of a derivative needs to be used. The definition of a derivative is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

The power rule verification of the function $f(x) = x^2$, whose derivative at $x = -2$ is $f'(-2) = -4$, would be

$f'(-2) = \lim_{h \to 0} \frac{(-2+h)^2 - 4}{h}$ at $x = -2$. Therefore, the alternative given does not represent a way to verify the power rule for that function.

13. A

According to the constant multiple rule, if $g(x) = kh(x)$, then $g'(x) = kh'(x)$. Therefore, the graph of $h'(x)$ in relation to the graph $g'(x)$ has a vertical stretch by a factor of k and a reflection on the x-axis (since $k < 0$). This graph is the only one that has a reflection on the x-axis.



14. WR

Step 1
Determine the derivative of the function $f(x) = 2x^3 - \frac{11}{2}x^2 + 5x$.

$$\begin{aligned} f(x) &= 2x^3 - \frac{11}{2}x^2 + 5x \\ f'(x) &= 6x^2 - 11x + 5 \end{aligned}$$

Step 2
The slope of the tangent is equal to $f'(x)$. Substitute 15 for $f'(x)$ into the function $f'(x) = 6x^2 - 11x + 5$, and solve for x.

$$\begin{aligned} f'(x) &= 6x^2 - 11x + 5 \\ 15 &= 6x^2 - 11x + 5 \\ 0 &= 6x^2 - 11x - 10 \\ 0 &= 6x^2 + 4x - 15x - 10 \\ 0 &= 2x(3x+2) - 5(3x+2) \\ 0 &= (2x-5)(3x+2) \end{aligned}$$

Therefore, $x = \frac{5}{2}$ and $x = -\frac{2}{3}$.

Step 3

Solve for $f\left(\frac{5}{2}\right)$.

$$\begin{aligned} f(x) &= 2x^3 - \frac{11}{2}x^2 + 5x \\ f\left(\frac{5}{2}\right) &= 2\left(\frac{5}{2}\right)^3 - \frac{11}{2}\left(\frac{5}{2}\right)^2 + 5\left(\frac{5}{2}\right) \\ f\left(\frac{5}{2}\right) &= \frac{125}{4} - \frac{275}{8} + \frac{25}{2} \\ f\left(\frac{5}{2}\right) &= 9.375 \end{aligned}$$

Practice TEST

Step 4

Solve for $f\left(-\frac{2}{3}\right)$.

$$f(x) = 2x^3 - \frac{11}{2}x^2 + 5x$$
$$f\left(-\frac{2}{3}\right) = 2\left(-\frac{2}{3}\right)^3 - \frac{11}{2}\left(-\frac{2}{3}\right)^2 + 5\left(-\frac{2}{3}\right)$$
$$f\left(-\frac{2}{3}\right) = -\frac{16}{27} - \frac{44}{18} - \frac{10}{3}$$
$$f\left(-\frac{2}{3}\right) \approx 6.37$$

When the coordinates are rounded to the nearest tenth, the slope of the tangent for the curve $f(x) = 2x^3 - \frac{11}{2}x^2 + 5x$ is equal to 15 at (2.5, 9.4) and (−0.7, 6.4).

15. A

Step 1

Determine the derivative of $f(x) = \sqrt{x}$.

$$f(x) = \sqrt{x}$$
$$f(x) = (x)^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$
$$f'(x) = \frac{1}{2\sqrt{x}}$$

Step 2

Substitute 4 for x into the function $f'(x) = \frac{1}{2\sqrt{x}}$, and solve for $f'(4)$.

$$f'(x) = \frac{1}{2\sqrt{x}}$$
$$f'(4) = \frac{1}{2\sqrt{4}}$$
$$f'(4) = \frac{1}{2(2)}$$
$$f'(4) = \frac{1}{4}$$

Therefore, the tangent line at $x = 4$ has a slope of $\frac{1}{4}$.

16. WR

Let $g(x) = e^{x^2}$ and $h(x) = \cos(2x + 1)$.

Step 1

Apply the chain rule to determine $g'(x)$ and $h'(x)$.

$$g(x) = e^{x^2}$$
$$g'(x) = e^{x^2}\left(\frac{d}{dx}x^2\right)$$
$$g'(x) = 2x\left(e^{x^2}\right)$$
$$h(x) = \cos(2x + 1)$$
$$h'(x) = -\sin(2x + 1)\left(\frac{d}{dx}(2x + 1)\right)$$
$$= -2\sin(2x + 1)$$

Step 2

Determine the derivative of the equation $f(x) = e^{x^2}\cos(2x + 1)$ by applying the product rule.

$$f(x) = g(x)h(x)$$
$$f'(x) = g'(x)h(x) + g(x)h'(x)$$
$$= \left(\begin{array}{l}\left(2xe^{x^2}\right)(\cos(2x + 1)) \\ + \left(e^{x^2}\right)(-2\sin(2x + 1))\end{array}\right)$$
$$= \left(\begin{array}{l}2xe^{x^2}\cos(2x + 1) \\ - 2e^{x^2}\sin(2x + 1)\end{array}\right)$$
$$= 2e^{x^2}[x\cos(2x + 1) - \sin(2x + 1)]$$

17. A

The slope at point B on the graph is positive, so $f'(x) > 0$. At point B, the graph is concave up, so $f''(x) > 0$.

18. WR

The graphs are as follows:

$f(x) = C$
$f'(x) = B$
$f''(x) = A$

When the slope of graph C is positive, function B is also positive. Likewise, when the slope of graph C is negative, function B is also negative. When the slope of graph C is zero, function B has a value of zero.

When the graph of C is concave up, function A is positive, and when the graph of C is concave down, function A is negative. At the point of inflection on graph C, function A has a value of zero.

19. WR

Step 1
Determine the second derivative of $f(x)$.

$f(x) = x^3 + ax^2$
$f'(x) = 3x^2 + 2ax$
$f''(x) = 6x + 2a$

Step 2
Determine the value of a.

A point of inflection occurs when $x = 6$ and $f''(x)$ is equal to zero.

Solve for a in the second derivative function when $x = 6$ and $f''(x) = 0$.

$$\begin{aligned} f''(x) &= 6x + 2a \\ 0 &= 6(6) + 2a \\ 0 &= 36 + 2a \\ -36 &= 2a \\ a &= -18 \end{aligned}$$

20. D

The local extrema of a function occur at values of x where $f'(x) = 0$.

Solve for x when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= 5x^4 + 20x^3 \\ 0 &= 5x^4 + 20x^3 \\ &= 5x^3(x + 4) \end{aligned}$$

Therefore, the x-coordinates of the local extrema of $f(x)$ are $x = 0$ and $x = -4$.

21. A

The graph of the derivative is a quadratic function, so the graph of the original function must be a cubic function. The graph of $f'(x)$ is equal to zero at $x = -1$ and $x = 1$. Therefore, the slope is equal to zero at these points on the graph of $f(x)$. When the slope is equal to zero at a particular point, there is a local maximum or minimum at that point.

Notice that the y-values to the left of $x = -1$ are positive, and the y-values to the right are negative. Hence, there is a local maximum at $x = -1$ on the graph of $f(x)$. The y values to the left of $x = 1$ are negative, and the y-values to the right are positive. Therefore, there is a local minimum at $x = 1$ on the graph of $f(x)$. Graph A corresponds to these characteristics.

22. D

Step 1
Solve $f'(x)$ for x when $f'(x) = 0$.

$$\begin{aligned} f'(x) &= x^2(x - 1) \\ 0 &= x^2(x - 1) \end{aligned}$$

$x^2 = 0$ or $x - 1 = 0$
$x = 0$ $\quad$ $x = 1$

Step 2
Test for intervals of increase and decrease using the first derivative function.

Interval	$f'(x)$	$f(x)$
$-\infty < x < 0$	$-$	Decreasing
$0 < x < 1$	$-$	Decreasing
$1 < x < \infty$	$+$	Increasing

Step 3
Test for intervals of concavity using the second derivative function.

Interval	$f''(x)$	$f(x)$
$-\infty < x < 0$	$+$	Concave up
$0 < x < 1$	$-$	Concave down
$1 < x < \infty$	$+$	Concave up

Therefore, the function $f(x)$ is both increasing and concave up on the interval $1 < x < \infty$.

23. C

A local maximum or minimum of a function occurs when the first derivative is equal to zero ($f'(x) = 0$). At a local maximum, the graph of a function is concave down when the second derivative is negative ($f''(x) < 0$).

24. D

When the second derivative of a polynomial function is zero when $x = a$, there is a point of inflection at $(a, f(a))$. Thus, there is a point of inflection at the point $\left(\frac{1}{2}, -\frac{1}{16}\right)$.

25. A

Step 1
Find the first derivative (velocity function) of the position function.

$s(t) = t^3 - 4.5t^2 - 7t$
$s'(t) = v(t) = 3t^2 - 9t - 7$

Step 2
Determine when the velocity of the particle reaches 5 m/s.
Set the velocity function equal to 5, and solve for t.

$$\begin{aligned} v(t) &= 3t^2 - 9t - 7 \\ 5 &= 3t^2 - 9t - 7 \\ 0 &= 3t^2 - 9t - 12 \\ 0 &= 3(t^2 - 3t - 4) \\ 0 &= 3(t-4)(t+1) \\ t &= 4, t = -1 \end{aligned}$$

Since t can only be a positive value, the velocity of the particle will reach 5 m/s when $t = 4.0$ s.

26. WR

Economic inflation refers to an increase in prices of goods over time.

In the graph provided, the inflation rate is positive for the entire 12-month period shown; therefore, prices were always increasing from January to December 2008. However, the curve-like nature of the graph suggests that while prices were increasing over the year, the rate of price increases per month varied significantly.

To determine the 4-month period in which prices were the most stable but still increasing, look for any 4-month period on the graph (since the prices were always increasing) where the inflation rate is lowest and decreasing. Over this period, the difference in price increases from one month to the next will be the smallest; therefore, the prices will be the most stable.

According to the given graph, prices were still increasing from September to December, but they were the most stable during this part of the year.

27. C

Step 1
Determine the function that represents the rate at which the pump removes the oil.
Find the derivative of function T.

$$T = \frac{250t^{\frac{3}{2}}}{3}$$

$$\begin{aligned} T' &= \left(\frac{250}{3}\right)\left(\frac{3}{2}\right)t^{\frac{1}{2}} \\ &= 125\sqrt{t} \end{aligned}$$

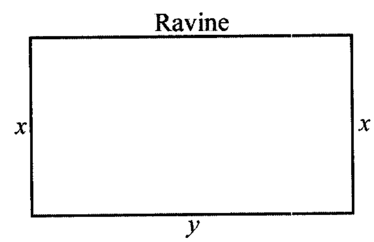
Step 2
Determine the amount of time it took for the pump to begin removing oil at a rate of 625 m^3/h.
Set the derivative function equal to 625, and solve for t.

$$\begin{aligned} T' &= 125\sqrt{t} \\ 625 &= 125\sqrt{t} \\ 5 &= \sqrt{t} \\ t &= 25 \end{aligned}$$

It will take 25 h for the pump to reach the point of removing oil at a rate of 625 m^3/h.

28. WR

Step 1
Sketch a diagram of the school yard.



Step 2
Determine the area formula in terms of one variable.
The perimeter of the yard requiring a fence is $P = 2x + y$. Since there are 300 m of available fencing, $300 = 2x + y$.
Isolate y in the perimeter formula.

$$\begin{aligned} 300 &= 2x + y \\ 300 - 2x &= y \end{aligned}$$

Substitute the expression for y in the area formula.

$$\begin{aligned} A &= xy \\ &= x(300 - 2x) \\ &= 300x - 2x^2 \end{aligned}$$

Step 3
Determine the derivative of A.

$$\begin{aligned} A &= 300x - 2x^2 \\ A' &= 300 - 4x \end{aligned}$$

Step 4
Determine the value of length x that maximizes the area.

Solve for x when $A' = 0$.

$$A' = 300 - 4x$$
$$0 = 300 - 4x$$
$$4x = 300$$
$$x = 75$$

Confirm the maximum as follows:

Interval	$A'(x)$	$A(x)$
$0 < x < 75$	+	Increasing
$75 < x < 150$	–	Decreasing

When the length, x, is 75 m, the area will be at a maximum.

Step 5
Determine the value of width y that maximizes the area.

Solve for y using the perimeter formula.

$$y = 300 - 2x$$
$$= 300 - 2(75)$$
$$= 150$$

Step 6
Determine the maximum area enclosed by 300 m of fencing.

$$A = xy$$
$$= 75(150)$$
$$= 11\ 250$$

The maximum area of a yard that can be enclosed by 300 m of fencing on three sides is 11 250 m^2.

29. WR

Step 1
Determine the function that represents the toy manufacturer's profit.

The manufacturer's monthly profit (P) is determined by the difference between its monthly revenue and monthly cost.

$$P(x) = R(x) - C(x)$$
$$= \left(\begin{array}{l}(20x - 0.002x^2) \\ -\ (38 + 12x + 0.002x^2)\end{array}\right)$$
$$= -0.004x^2 + 8x - 38$$

Step 2
Determine the derivative of the profit function.

$$P(x) = -0.004x^2 + 8x - 38$$
$$P'(x) = -0.008x + 8$$

Step 3
Determine how many dolls must be sold to maximize profit.

Set $P'(x)$ equal to zero, and solve for x.

$$P'(x) = -0.008x + 8$$
$$0 = -0.008x + 8$$
$$x = 1\ 000$$

Step 4
Confirm the maximum.

Interval	$P'(x)$	$P(x)$
$0 < x < 1\ 000$	+	Increasing
$1\ 000 < x < 2\ 000$	–	Decreasing

For values of x between 0 and 1 000, the first derivative of the profit function is positive; for values between 1 000 and 2 000, the first derivative is negative. This means there is a maximum when $x = 1\ 000$.

Therefore, the toy manufacturer must sell 1 000 dolls to maximize profits.

30. C

Step 1
Determine an exponential function that models the growth of the rabbit population.

The population of rabbits is given by $C(t) = Ae^{kt}$. At $t = 0$, there are A rabbits. After 2.5 years, the population of rabbits doubles.

$$C(t) = Ae^{kt}$$
$$C(0) = A$$
$$C(2.5) = 2A$$

Solve for k.

$$C(t) = Ae^{kt}$$
$$2A = Ae^{2.5k}$$
$$2 = e^{2.5k}$$
$$\ln 2 = 2.5k$$
$$\frac{1}{2.5}\ln 2 = k$$

Therefore, the exponential function to work with is $C(t) = Ae^{\ln 2\left(\frac{t}{2.5}\right)}$.

Step 2
Determine when the population of rabbits will triple.
When $C(t) = 3A$, the population will have tripled.

$$C(t) = Ae^{\ln 2\left(\frac{t}{2.5}\right)}$$
$$3A = Ae^{\ln 2\left(\frac{t}{2.5}\right)}$$
$$3 = e^{\ln 2\left(\frac{t}{2.5}\right)}$$
$$\ln 3 = \ln 2\left(\frac{t}{2.5}\right)$$
$$\frac{2.5\ln 3}{\ln 2} = t$$
$$3.96 \approx t$$

After 3.96 years, the population of rabbits will have tripled.

Step 3
Determine the rate of growth function.
Find the first derivative of $C(t)$.

$$C(t) = Ae^{\ln 2\left(\frac{t}{2.5}\right)}$$

$$C'(t) = Ae^{\ln 2\left(\frac{t}{2.5}\right)} \cdot \ln 2\left(\frac{1}{2.5}\right)$$

Step 4
Determine the rate of growth when the population triples.
Solve for $C'(t)$ when $t = 3.96$.

$$C'(t) = Ae^{\ln 2\left(\frac{t}{2.5}\right)} \cdot \ln 2\left(\frac{1}{2.5}\right)$$

$$C'(3.96) = Ae^{\ln 2\left(\frac{3.96}{2.5}\right)} \cdot \ln 2\left(\frac{1}{2.5}\right)$$

$$\approx 0.83A$$

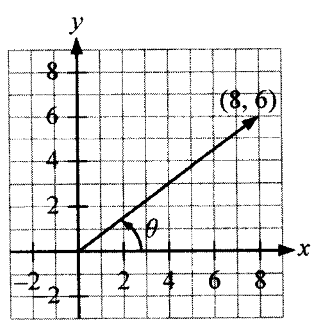
Therefore, the rate of growth of the population of rabbits will be approximately $0.83A$ rabbits/a at the instant when the population of rabbits triples.

31. C

Equal geometric vectors must have the same magnitude (length) and direction. The three parallel vectors pointed to the upper right appear to be the same length and have the same direction.

32. A

Step 1
Sketch a diagram of the vector.



Step 2
Determine the magnitude r.
Use the Pythagorean theorem.

$$\begin{aligned} r &= \sqrt{x^2 + y^2} \\ &= \sqrt{8^2 + 6^2} \\ &= \sqrt{100} \\ &= 10 \end{aligned}$$

Step 3
Determine the rotational angle θ.
Use the tangent ratio.

$$\begin{aligned} \tan\theta &= \frac{y}{x} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ &= \tan^{-1}\left(\frac{6}{8}\right) \\ &\approx 36.9^\circ \end{aligned}$$

The polar form is approximately $(10, 36.9^\circ)$.

33. 5

Step 1
Determine $2\vec{u}$.

$$\begin{aligned} 2\vec{u} &= 2(2, -4, 6) \\ &= (2(2), 2(-4), 2(6)) \\ &= (4, -8, 12) \end{aligned}$$

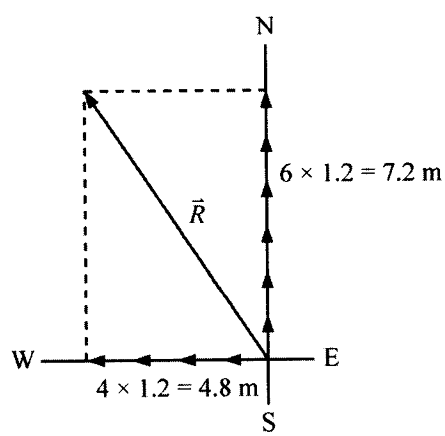
Step 2
Substitute $(4, -8, 12)$ for $2\vec{u}$ and (a, b, c) for $\vec{v}$ into the equation $(8, -3, 5) = 2\vec{u} + \vec{v}$.

$$\begin{aligned} (8, -3, 5) &= 2\vec{u} + \vec{v} \\ (8, -3, 5) &= (4, -8, 12) + (a, b, c) \\ (a, b, c) &= (8, -3, 5) - (4, -8, 12) \\ (a, b, c) &= (8 - 4, -3 - (-8), 5 - 12) \\ (a, b, c) &= (4, 5, -7) \end{aligned}$$

Therefore, the value of b is 5.

34. a) WR

Step 1
Draw a vector diagram of the frog's movements and the resultant displacement, $\vec{R}$.



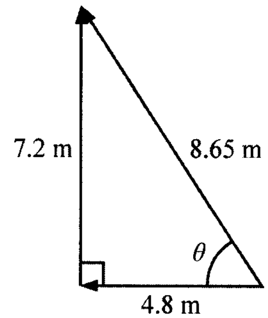
Step 2
Determine the magnitude of the resultant, $\vec{R}$, to the nearest hundredth metre using the Pythagorean theorem.

$$c^2 = a^2 + b^2$$
$$|\vec{R}|^2 = 4.8^2 + 7.2^2$$
$$|\vec{R}| = \sqrt{4.8^2 + 7.2^2}$$
$$|\vec{R}| \approx 8.65 \text{ m}$$

Therefore, the frog is approximately 8.65 m from its original position after 10 hops.

b) WR

Step 1
Draw a sketch of the frog's displacements, indicating the direction of the resultant with angle θ.



Step 2
Determine the measure of angle θ to the nearest tenth of a degree.
Since the displacements form a right triangle, use a trigonometric ratio to find the measure of angle θ.

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$
$$= \frac{7.2}{4.8}$$
$$= 1.5$$
$$\theta = \tan^{-1}(1.5)$$
$$\approx 56.3°$$

Note that $\sin\theta$ or $\cos\theta$ could have been used to find the measure of angle θ.

Step 3
Determine the bearing of the frog's resultant displacement.
A bearing is a direction clockwise from north. Therefore, the bearing of the frog's resultant displacement is 270° + 56.3° = 326.3°.

35. 39.5

Determine the value of c by using the definition of work as a dot product.

$$W = \vec{F} \cdot \vec{d}$$
$$7\,200 = (200, 500, 200) \cdot (4, -3, c)$$
$$7\,200 = 200(4) + 500(-3) + 200c$$
$$7\,200 = 800 - 1\,500 + 200c$$
$$7\,200 = -700 + 200c$$
$$200c = 7\,900$$
$$c = 39.5$$

36. D

Step 1
Since equations A and B are based on the definitions of the dot and cross products between any two vectors, they can be used to find the measure of θ.

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos\theta \text{ or } \cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$
$$|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta \text{ or } \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

By applying these definitions to the vectors $\vec{a} = (4, 0, 0)$ and $\vec{b} = (4, 3, 5)$, the value of θ can be determined.

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$
$$\cos\theta = \frac{(4)(4) + (0)(3) + (0)(5)}{\sqrt{4^2 + 0^2 + 0^2}\sqrt{4^2 + 3^2 + 5^2}}$$
$$\cos\theta = \frac{16}{\sqrt{16}\sqrt{16 + 9 + 25}}$$
$$\cos\theta = \frac{16}{4\sqrt{50}}$$
$$\theta = \cos^{-1}\frac{16}{4\sqrt{50}}$$
$$\theta \approx 55.55°$$

Step 2
Compare the value of θ to the value related to equation C.

$$\tan\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}|^2}$$
$$= \frac{\sqrt{544}}{(4)^2}$$
$$= \frac{4\sqrt{34}}{16}$$
$$= \frac{\sqrt{34}}{4}$$
$$\theta = \tan^{-1}\frac{\sqrt{34}}{4}$$
$$\theta \approx 55.55°$$

Step 3
Compare the value of θ to the value related to equation D.

$$\cot\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$$
$$= \frac{16}{(\sqrt{50})^2}$$
$$= \frac{16}{50}$$
$$\frac{1}{\tan\theta} = \frac{16}{50}$$
$$\tan\theta = \frac{50}{16}$$
$$\theta = \tan^{-1}\frac{50}{16}$$
$$\theta \approx 72.26°$$

The value of θ determined from equation C is correct, but the value of θ determined from equation Dis incorrect. Therefore, $\cot\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}$ cannot be used to find the measure of the angle, θ, between the two given vectors.

37. D

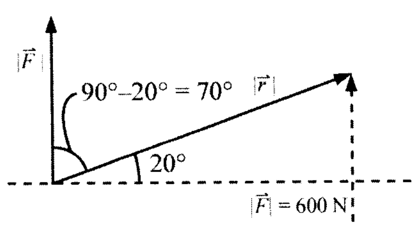
Check to see if the vectors $(5\sqrt{2}, 2, -2\sqrt{2})$ and $(5, \sqrt{2}, -2)$ are collinear.
$(5\sqrt{2}, 2, -2\sqrt{2}) = k(5, \sqrt{2}, -2)$
$(5\sqrt{2}, 2, -2\sqrt{2}) = (5k, k\sqrt{2}, -2k)$

- $5k = 5\sqrt{2}$
  $k = \sqrt{2}$
- $k\sqrt{2} = 2$
  $k = \sqrt{2}$
- $-2k = -2\sqrt{2}$
  $k = \sqrt{2}$

Vectors $(5\sqrt{2}, 2, -2\sqrt{2})$ and $(5, \sqrt{2}, -2)$ are collinear vectors. The cross product of collinear vectors is the zero vector, (0, 0, 0).

38. C

Translate vector $\vec{F}$ so that it is in a tail-to-tail arrangement with vector $\vec{r}$. The measure of the angle formed between $\vec{F}$ and $\vec{r}$ is then 90° – 20° = 70°.



Since the magnitude of torque is defined by $|\vec{\tau}| = |\vec{r}||\vec{F}|\sin\theta$, the correct equation representing the torque is $|\vec{\tau}| = (0.045)(600)\sin 70°$.

39. WR

Step 1
Rewrite the equation of each line in the slope-intercept form $y = mx + b$.

l_1: $3x + y - 2 = 0$
$y = -3x + 2$

l_2: $2Ax - 3By + 24 = 0$
$-3By = -2Ax - 24$
$y = \frac{2A}{3B}x + \frac{24}{3B}$

Step 2
In order for the lines to be coincident, their y-intercepts (b) and slopes (m) must be equal.
Solve for B.
Equate the y-intercepts from l_1 and l_2.

$$2 = \frac{24}{3B}$$
$$6B = 24$$
$$B = 4$$

Step 3
Solve for A.
Equate the slopes from l_1 and l_2.

$$-3 = \frac{2A}{3B}$$
$$-3 = \frac{2A}{3(4)}$$
$$-36 = 2A$$
$$-18 = A$$

When $A = -18$ and $B = 4$, the lines are coincident.

40. D

Step 1
Rewrite the equation in the form $Ax + By + Cz + D = 0$.
$2x - y - 8 = 0$
$2x - y + 0z - 8 = 0$

Step 2
Substitute the coordinates (2, –4, 0) in for $x, y,$ and z.

$$2x - y + 0z - 8 = 0$$
$$2(2) - (-4) + 0(0) - 8 = 0$$
$$4 + 4 - 8 = 0$$
$$0 = 0$$

Step 3
Substitute the coordinates (0, –8, 8) in for $x, y,$ and z.

$$2x - y + 0z - 8 = 0$$
$$2(0) - (-8) + 0(8) - 8 = 0$$
$$8 - 8 = 0$$
$$0 = 0$$

Step 4
Substitute the coordinates (4, 0, −2) in for x, y, and z.

$$\begin{aligned} 2x - y + 0z - 8 &= 0 \\ 2(4) - (0) + 0(-2) - 8 &= 0 \\ 8 - 8 &= 0 \\ 0 &= 0 \end{aligned}$$

Step 5
Substitute the coordinates (−4, 0, 5) in for x, y, and z.

$$\begin{aligned} 2x - y + 0z - 8 &= 0 \\ 2(-4) - (0) + 0(5) - 8 &= 0 \\ -8 - 8 &= 0 \\ -16 &\neq 0 \end{aligned}$$

Therefore, the point (−4, 0, 5) does not lie on the plane defined by the equation $2x - y - 8 = 0$.

41. a) WR

Step 1
The x-intercept has the coordinates $(x, 0)$. Set $y = 0$, and solve for t.

$$\begin{aligned} y &= -\sqrt{2} + 2\sqrt{3}t \\ 0 &= -\sqrt{2} + 2\sqrt{3}t \\ \sqrt{2} &= 2\sqrt{3}t \\ \frac{\sqrt{2}}{2\sqrt{3}} &= t \end{aligned}$$

Step 2
Substitute $\frac{\sqrt{2}}{2\sqrt{3}}$ for t into the equation $x = \sqrt{2} - \sqrt{3}t$ to solve for the x-intercept.

$$\begin{aligned} x &= \sqrt{2} - \sqrt{3}t \\ &= \sqrt{2} - \sqrt{3}\left(\frac{\sqrt{2}}{2\sqrt{3}}\right) \\ &= \sqrt{2} - \frac{\sqrt{2}}{2} \\ &= \frac{2\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

The x-intercept is $\frac{\sqrt{2}}{2}$ or $\left(\frac{\sqrt{2}}{2}, 0\right)$.

b) WR

Step 1
Isolate each part for t.

$$\begin{aligned} x &= \sqrt{2} - \sqrt{3}t \\ x - \sqrt{2} &= -\sqrt{3}t \\ \frac{x - \sqrt{2}}{-\sqrt{3}} &= t \end{aligned}$$

$$\begin{aligned} y &= -\sqrt{2} + 2\sqrt{3}t \\ y + \sqrt{2} &= 2\sqrt{3}t \\ \frac{y + \sqrt{2}}{2\sqrt{3}} &= t \end{aligned}$$

Step 2
Equate the expressions of t.

$$\begin{aligned} t &= t \\ \frac{x - \sqrt{2}}{-\sqrt{3}} &= \frac{y + \sqrt{2}}{2\sqrt{3}} \\ 2\sqrt{3}(x - \sqrt{2}) &= -\sqrt{3}(y + \sqrt{2}) \\ 2\sqrt{3}x - 2\sqrt{6} &= -\sqrt{3}y - \sqrt{6} \\ 2\sqrt{3}x + \sqrt{3}y - 2\sqrt{6} + \sqrt{6} &= 0 \\ 2\sqrt{3}x + \sqrt{3}y - \sqrt{6} &= 0 \end{aligned}$$

The equation of the line in scalar form is $2\sqrt{3}x + \sqrt{3}y - \sqrt{6} = 0$.

42. C

Step 1
Add P_2 to P_1 to eliminate z.

$$\begin{aligned} P_1&: 3x - 2y + z - 5 = 0 \\ P_2&: \underline{x - 2y - z + 2 = 0} \\ &\quad 4x - 4y + 0z - 3 = 0 \end{aligned}$$

Step 2
Substitute each parametric equation of x, y, and z into the equation of the line to see which one defines the line.

For A, the substitution is as follows:

$$\begin{aligned} 4x - 4y + 0z - 3 &= 0 \\ 4(t) - 4(t - 3) + 0\left(-t + \frac{7}{2}\right) - 3 &= 0 \\ 4t - 4t + 12 + 0 - 3 &= 0 \\ 9 &= 0 \end{aligned}$$

For B, the substitution is as follows:

$$\begin{aligned} 4x - 4y + 0z - 3 &= 0 \\ 4(t) - 4\left(t + \frac{3}{4}\right) + 0(-t - 5) - 3 &= 0 \\ 4t - 4t - 3 + 0 - 3 &= 0 \\ -6 &= 0 \end{aligned}$$

For C, the substitution is as follows:

$$\begin{aligned} 4x - 4y + 0z - 3 &= 0 \\ 4\left(t + \frac{3}{4}\right) - 4(t) + 0\left(-t + \frac{11}{4}\right) - 3 &= 0 \\ 4t + 3 - 4t + 0 - 3 &= 0 \\ 0 &= 0 \end{aligned}$$

For D, the substitution is as follows:

$$\begin{aligned} 4x - 4y + 0z - 3 &= 0 \\ 4\left(\frac{7}{2} - t\right) - 4(-t + 11) + 0(t) - 3 &= 0 \\ 14 - 4t + 4t - 44 - 3 &= 0 \\ -33 &= 0 \end{aligned}$$

When the parametric equations in Care substituted into the equation of the line and simplified, the left side of the equation equals the right side of the equation. Therefore, the parametric equations given in C define the line.

Practice TEST

43. B

A vector, $\vec{v}$, that is perpendicular to a plane must be parallel to the normal vector, $\vec{n}$, which means that $\vec{v} = k\vec{n}$, where $k \neq 0$.

Step 1
Determine a normal vector, $\vec{n}$, to the plane $\frac{3}{4}x - \frac{1}{2}y + 3z - 8 = 0$.

A normal vector to the plane is $\vec{n} = \left(\frac{3}{4}, -\frac{1}{2}, 3\right)$.

Step 2
Determine which of the given vectors is parallel to $\vec{n}$.

Alternative A:

$$\vec{u} = k\vec{n}$$
$$\left(-\frac{2}{3}, 1, -6\right) = k\left(\frac{3}{4}, -\frac{1}{2}, 3\right)$$

$-\frac{2}{3} = \frac{3}{4}k$ $k = -\frac{8}{9}$	$1 = -\frac{1}{2}k$ $k = -2$	$-6 = 3k$ $k = -2$

Since the k-value is not constant, $\vec{u}$ is not perpendicular to the plane.

Alternative B:

$$\vec{v} = k\vec{n}$$
$$\left(1, -\frac{2}{3}, 4\right) = k\left(\frac{3}{4}, -\frac{1}{2}, 3\right)$$

$1 = \frac{3}{4}k$ $k = \frac{4}{3}$	$-\frac{2}{3} = -\frac{1}{2}k$ $k = \frac{4}{3}$	$4 = 3k$ $k = \frac{4}{3}$

Since the k-value is constant, $\vec{v}$ is perpendicular to the plane.

Alternative C:

$$\vec{w} = k\vec{n}$$
$$(4, 0, -1) = k\left(\frac{3}{4}, -\frac{1}{2}, 3\right)$$

$4 = \frac{3}{4}k$ $k = \frac{16}{3}$	$0 = -\frac{1}{2}k$ $k = 0$	$-1 = 3k$ $k = -\frac{1}{3}$

Since the k-value is not constant, $\vec{w}$ is not perpendicular to the plane.

Alternative D:

$$\vec{x} = k\vec{n}$$
$$(8, 2, 1) = k\left(\frac{3}{4}, -\frac{1}{2}, 3\right)$$

$8 = \frac{3}{4}k$ $k = \frac{32}{3}$	$2 = -\frac{1}{2}k$ $k = -4$	$1 = 3k$ $k = \frac{1}{3}$

Since the k-value is not constant, $\vec{x}$ is not perpendicular to the plane.

44. A

Step 1
Find two direction vectors defining the plane.
Vector $\vec{u} = (3, -2, 1)$ is a direction vector since it is the direction vector of the line contained by the plane. Another non-parallel direction vector, $\vec{v}$, can be found by using the point $(1, 0, 2)$ of the line and the y-intercept $(0, 5, 0)$.

$$\vec{v} = (0, 5, 0) - (1, 0, 2) = (-1, 5, -2)$$

Step 2
Find a normal vector, $\vec{n}$, to the plane.

$$\begin{aligned}\vec{n} &= \vec{u} \times \vec{v} \\ &= (3, -2, 1) \times (-1, 5, -2) \\ &= \begin{bmatrix}(-2)(-2)-(1)(5), \\ (1)(-1)-(3)(-2), \\ (3)(5)-(-2)(-1)\end{bmatrix} \\ &= (-1, 5, 13)\end{aligned}$$

Step 3
Find the scalar equation of the plane.
Since $\vec{n} = (-1, 5, 13)$, the scalar equation can be defined as $-x + 5y + 13z + D = 0$.
The value of D can be found by using one of the points on the plane, such as the y-intercept $(0, 5, 0)$.

$$\begin{aligned}-x + 5y + 13z + D &= 0 \\ -(0) + 5(5) + 13(0) + D &= 0 \\ 25 + D &= 0 \\ D &= -25\end{aligned}$$

Therefore, the scalar equation of the plane is $-x + 5y + 13z - 25 = 0$.

45. B

The vector form of a plane consists of a position vector that is represented by a point on the plane. In this case, it could be the x-intercept, $\vec{u}$, or the z-intercept, $\vec{v}$. The vector form of a plane also consists of two non-parallel direction vectors. One of these could be given by the difference between the x-intercept and z-intercept, that is, by $\vec{v} - \vec{u}$ or $\vec{u} - \vec{v}$. The other direction vector could be defined by a point that is not collinear with an x-intercept or z-intercept. More precisely, this would be a point with a y-coordinate. Equation B is the only one with such a direction vector. The other equations have a second direction vector collinear with $\vec{v} - \vec{u}$.

46. WR

Step 1
Write the equation of the line in parametric form.

$$L_1: \begin{aligned} x &= -1 + 2s \\ y &= 4 + s \\ z &= -2 - 3s\end{aligned}$$

Step 2
Substitute the expressions for x, y, and z into the equation of the plane.

$$\begin{aligned}P_1: 0 &= -2x + y + 3z + 6 \\ 0 &= \begin{pmatrix}-2(-1+2s)+(4+s) \\ +3(-2-3s)+6\end{pmatrix} \\ 0 &= 2 - 4s + 4 + s - 6 - 9s + 6 \\ 0 &= -12s + 6 \\ 12s &= 6 \\ s &= \frac{1}{2}\end{aligned}$$

Step 3
Substitute $\frac{1}{2}$ for s into the equation of the line, L_1, and find the point of intersection.

$$\begin{aligned}L_1: (x, y, z) &= (-1, 4, -2) + s(2, 1, -3) \\ &= (-1, 4, -2) + \frac{1}{2}(2, 1, -3) \\ &= (-1, 4, -2) + \left(1, \frac{1}{2}, -\frac{3}{2}\right) \\ &= \left(0, \frac{9}{2}, -\frac{7}{2}\right)\end{aligned}$$

The point of intersection between the line and the plane is $\left(0, \frac{9}{2}, -\frac{7}{2}\right)$.

47. C

Three planes can be consistent by having one solution (when they intersect at one point) or infinite solutions (when they intersect at a line or as one plane). Three planes can also be inconsistent with no solution between them (e.g., three distinct parallel planes, two distinct parallel planes with one plane intersecting them, and so on). It is impossible for three planes to pass through two distinct points (to have two solutions).

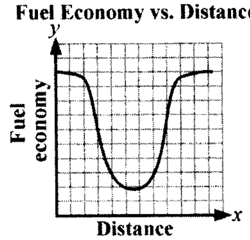
ANSWERS AND SOLUTIONS — PRACTICE TEST 2

1. A	11. D	21. WR	31. 5	40. A
2. D	12. B	22. D	32. D	41. WR
3. 3.5	13. D	23. C	33. 11.2	42. B
4. D	14. C	24. C	34. WR	43. A
5. 4.01	15. WR	25. 1.25	35. B	44. WR
6. B	16. WR	26. C	36. a) WR	45. WR
7. A	17. C	27. B	b) WR	46. WR
8. WR	18. B	28. WR	37. WR	
9. 3.4	19. C	29. C	38. C	
10. WR	20. WR	30. B	39. C	

1. A

The higher the fuel consumption is, the lower the fuel economy will be. As the car drives at the bottom of the valley, the rate of fuel consumption is low; therefore, the fuel economy is high. As the car drives up the hill, the rate of fuel consumption becomes higher as the incline of the hill increases; therefore, the fuel economy becomes lower. The incline of the hill will eventually hit its steepest part, at which point the rate of fuel consumption will be the highest; therefore, the fuel economy will be at its lowest. The hill then begins to flatten out, lowering the rate of fuel consumption; therefore, the fuel economy becomes higher. Eventually, the road will become flat again, as it was at the bottom of the valley, bringing the rate of fuel consumption back to its original value.

This graph shows Mr. Green's fuel economy.



2. D

Determine the average rate of change over each interval by calculating the slope, m, between the points.

Step 1
Calculate the slope between $x = 0$ and $x = 1$.

$$m = \frac{5-4}{1-0} = \frac{1}{1} = 1$$

Step 2
Calculate the slope between $x = 5$ and $x = 10$.

$$m = \frac{50-25}{10-5} = \frac{25}{5} = 5$$

Step 3
Calculate the slope between $x = 0$ and $x = 23$.

$$m = \frac{200-4}{23-0} = \frac{196}{23} \approx 8.5$$

Step 4
Calculate the slope between $x = 16$ and $x = 23$.

$$m = \frac{200-110}{23-16} = \frac{90}{7} \approx 12.9$$

According to the slope values, the average rate of change is greatest from $x = 16$ to $x = 23$.

3. 3.5

Step 1

Divide the numerator and denominator by x^2, and simplify.

$$\lim_{x\to\pm\infty}\frac{7x^2+4}{2x^2-8}$$

$$=\lim_{x\to\pm\infty}\frac{\frac{7x^2}{x^2}+\frac{4}{x^2}}{\frac{2x^2}{x^2}-\frac{8}{x^2}}$$

$$=\lim_{x\to\pm\infty}\frac{7+\frac{4}{x^2}}{2-\frac{8}{x^2}}$$

Step 2

Evaluate the limit.

As $x\to\pm\infty$, the values of $\frac{4}{x^2}$ and $\frac{8}{x^2}$ approach 0.

$$\lim_{x\to\pm\infty}\frac{7+\frac{4}{x^2}}{2-\frac{8}{x^2}}=\frac{7+0}{2-0}$$

$$=3.5$$

4. D

The expression $\lim_{h\to0}\frac{f(a+h)-f(a)}{h}$ describes the instantaneous rate of change at $x=a$; i.e., the slope of the tangent at $x=a$. Since $x=-3$ in the expression $\lim_{h\to0}\frac{f(-3+h)-f(-3)}{h}=0$, the graph of $f(x)$ has a tangent at $x=-3$ that has a slope equal to 0 (which makes the tangent horizontal). Therefore, the graph of $f(x)$ has a horizontal tangent at $x=-3$.

5. 4.01

For the values $x=3.01$ and $x=3$, evaluate $\frac{f(x+h)-f(x)}{h}$, where $h=0.01$, in order to approximate the instantaneous rate of change at $x=3$ for $f(x)=x^2-2x+5$.

$$\frac{f(x+h)-f(x)}{h}$$

$$=\frac{f(3+0.01)-f(3)}{h}$$

$$=\frac{f(3.01)-f(3)}{h}$$

$$=\frac{\left((3.01)^2-2(3.01)+5\right)-\left((3)^2-2(3)+5\right)}{0.01}$$

$$=\frac{(9.0601-6.02+5)-(9-6+5)}{0.01}$$

$$=\frac{8.0401-8}{0.01}$$

$$=\frac{0.0401}{0.01}$$

$$=4.01$$

Based on $x=3.01$, the instantaneous rate of change is 4.01.

6. B

The instantaneous rate of change of a function is positive where the function is increasing when viewed from left to right, and it is negative where the function is decreasing when viewed from left to right. The instantaneous rate of change of a function is zero where the graph reaches a maximum or minimum.

In the given graph, the instantaneous rate of change for the function f is negative on $x<-5$, positive on $x>3$, zero at $x=-3$, and positive at $x=2$.

7. A

Simplify the expression using algebra.

$$\lim_{h\to0}\frac{2(x+h)^2-6(x+h)-\left(2x^2-6x\right)}{h}$$

$$=\lim_{h\to0}\frac{2\left(x^2+2xh+h^2\right)-6x-6h-2x^2+6x}{h}$$

$$=\lim_{h\to0}\frac{2x^2+4xh+2h^2-6x-6h-2x^2+6x}{h}$$

$$=\lim_{h\to0}\frac{4xh+2h^2-6h}{h}$$

$$=\lim_{h\to0}\frac{h(4x+2h-6)}{h}$$

$$=\lim_{h\to0}4x+2h-6$$

$$=4x-6$$

8. WR

The derivative of $f(x) = a^x$ is $f'(x) = a^x \ln a$.

For the function $f(x) = \left(\frac{1}{3}\right)^x$, the derivative can be determined as follows:

$$f(x) = \left(\frac{1}{3}\right)^x$$
$$f'(x) = \left(\frac{1}{3}\right)^x \ln\left(\frac{1}{3}\right)$$
$$f'(x) \approx -1.0986\left(\frac{1}{3}\right)^x$$

Therefore, the graph of $f(x)$ must undergo a vertical stretch with respect to the x-axis by a factor of $\left|\ln\left(\frac{1}{3}\right)\right| \approx 1.0986$ and a reflection in the x-axis.

9. 3.4

Step 1
Take the derivative of $f(x) = e^x$ and set the derivative to 29.8.

The derivative of $f(x) = e^x$ is $f'(x) = e^x$.

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$29.8 = e^x$$

Step 2
Take the natural logarithm of both sides of the equation.

$$29.8 = e^x$$
$$\ln(29.8) = \ln\left(e^x\right)$$

Step 3
Simplify and solve for x.

$$\ln(29.8) = \ln\left(e^x\right)$$
$$\ln(29.8) = x$$
$$3.4 \approx x$$

10. WR

Step 1
Convert the equation $\ln(\ln x) = 1$ into an exponential equation with a base of e.

$$\ln(\ln x) = 1$$
$$e^{\ln(\ln x)} = e^1$$
$$\ln x = e$$

Step 2
Convert the equation $\ln x = e$ into an exponential equation with a base of e.

$$\ln x = e$$
$$e^{\ln x} = e^e$$
$$x = e^e$$

11. D

The instantaneous rate of change of a function, $f(x)$, is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Therefore, the instantaneous rate of change for the function $f(x) = \left(\frac{1}{2}\right)^x$ can be expressed as

$$f'(x) = \lim_{h \to 0} \frac{\left(\frac{1}{2}\right)^{x+h} - \left(\frac{1}{2}\right)^x}{h}.$$

Since $\left(\frac{1}{2}\right)^x$ can also be expressed as 2^{-x}, the instantaneous rate of change of $f(x) = 2^{-x}$ can also be expressed by the following function:

$$f'(x) = \lim_{h \to 0} \frac{2^{-(x+h)} - 2^{-x}}{h}$$
$$= \lim_{h \to 0} \frac{2^{-x-h} - 2^{-x}}{h}$$

12. B

The power rule applies to functions that have a power, a base of x, and an exponent with a numerical value. The power rule does not apply to the function $f(x) = 2^x$. The function $f(x) = 2^x$ is an exponential function, and its derivative is $f'(x) = 2^x \ln 2$.

13. D

The sum rule states that $f'(x) + g'(x) = (f+g)'(x)$. When verifying the sum rule, the definition of a derivative can be used. The definition of a derivative is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Use this formula to find $f'(x)$, $g'(x)$, and $(f+g)'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{-2(x+h) + 2x}{h}$$
$$g'(x) = \lim_{h \to 0} \frac{-5(x+h)^3 + 5x^3}{h}$$
$$(f+g)'(x) = \lim_{h \to 0} \frac{\left[-2(x+h) - 5(x+h)^3\right] + 2x + 5x^3}{h}$$

Substitute the equations into the sum rule.

$$\lim_{h \to 0} \frac{-2(x+h) + 2x}{h} + \lim_{h \to 0} \frac{-5(x+h)^3 + 5x^3}{h} = \lim_{h \to 0} \frac{\left[-2(x+h) - 5(x+h)^3\right] + 2x + 5x^3}{h}$$

Practice TEST

14. C

Step 1
Substitute -1 for x in the function $f(x)=4x^3-5x+3$, and solve for $f(-1)$.

$$\begin{aligned} f(x) &= 4x^3-5x+3 \\ f(-1) &= 4(-1)^3-5(-1)+3 \\ f(-1) &= -4+5+3 \\ f(-1) &= 4 \end{aligned}$$

Step 2
Determine the derivative of the function $f(x)=4x^3-5x+3$.

$$\begin{aligned} f(x) &= 4x^3-5x+3 \\ f'(x) &= 12x^2-5 \end{aligned}$$

Step 3
Substitute -1 for x in the function $f'(x)=12x^2-5$, and solve for $f'(-1)$.

$$\begin{aligned} f'(x) &= 12x^2-5 \\ f'(-1) &= 12(-1)^2-5 \\ f'(-1) &= 12(1)-5 \\ f'(-1) &= 7 \end{aligned}$$

Step 4
Substitute -1 for x, 4 for y, and 7 for m in the equation $y=mx+b$, and solve for b.

$$\begin{aligned} y &= mx+b \\ 4 &= 7(-1)+b \\ 4 &= -7+b \\ 11 &= b \end{aligned}$$

Therefore, the equation of the tangent line is $y=7x+11$.

15. WR

To verify the chain rule for the function $f(x)=\left(-4x^{27}\right)^{\frac{1}{3}}$, it can be put into another form so the derivative can be taken using a method other than the chain rule. In this case, simplify the exponents.

$$\begin{aligned} f(x) &= \left(-4x^{27}\right)^{\frac{1}{3}} \\ &= (-4)^{\frac{1}{3}}\left(x^{27}\right)^{\frac{1}{3}} \\ &= -4^{\frac{1}{3}}x^9 \end{aligned}$$

The power rule can be used on the function $f(x)=-4^{\frac{1}{3}}x^9$ to verify the chain rule for the function $f(x)=\left(-4x^{27}\right)^{\frac{1}{3}}$.

16. WR

Step 1

Change the function $y=\dfrac{x^3}{\cos x}$ into product form.

$$\begin{aligned} y &= \frac{x^3}{\cos x} \\ y &= x^3(\cos x)^{-1} \end{aligned}$$

Step 2
Let $g(x)=x^3$ and $h(x)=(\cos x)^{-1}$, and determine $g'(x)$ and $h'(x)$.

$$\begin{aligned} g(x) &= x^3 \\ g'(x) &= 3x^2 \\ h(x) &= (\cos x)^{-1} \\ h'(x) &= -1(\cos x)^{-2}\left(\frac{d}{dx}\cos x\right) \\ &= -(\cos x)^{-2}(-\sin x) \\ &= \sin x(\cos x)^{-2} \end{aligned}$$

Step 3
Use the product rule to determine the derivative of $y=x^3(\cos x)^{-1}$.

$$\begin{aligned} y &= g(x)h(x) \\ \frac{dy}{dx} &= g'(x)h(x)+g(x)h'(x) \\ &= \left(3x^2\right)(\cos x)^{-1}+\left(x^3\right)\sin x(\cos x)^{-2} \\ &= \frac{3x^2}{\cos x}+\frac{x^3\sin x}{\cos^2 x} \end{aligned}$$

Step 4
Substitute π for x into the derivative $\dfrac{dy}{dx}=\dfrac{3x^2}{\cos x}+\dfrac{x^3\sin x}{\cos^2 x}$, and solve for $\dfrac{dy}{dx}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x^2}{\cos x}+\frac{x^3\sin x}{\cos^2 x} \\ &= \frac{3\pi^2}{\cos\pi}+\frac{\pi^3\sin\pi}{(\cos\pi)^2} \\ &= \frac{3\pi^2}{-1}+\frac{\pi^3(0)}{(-1)^2} \\ &= -3\pi^2+0 \\ &\approx -29.6 \end{aligned}$$

Therefore, the slope of the tangent line of the curve $y=\dfrac{x^3}{\cos x}$ at $x=\pi$ is $-3\pi^2$, or approximately -29.6.

17. C

The slope of the graph of $f(x)$ is zero at $x=-2$, 0 and 2. Therefore, the derivative graph must be equal to zero at these points.

Notice that the slope of the graph is positive to the left and negative to the right of $x = -2, 2$. Similarly, the slope of the graph is negative to the left and positive to the right of $x = 0$.

Therefore, the derivative graph, $f'(x)$, is positive on the intervals $x < -2$ and $0 < x < 2$ and negative on the intervals $-2 < x < 0$ and $x > 2$.

Graph C best represents these characteristics.

Note that the graph of $f(x)$ is a quartic function, so the graph of the derivative, $f'(x)$, must be a cubic function. Therefore, graphs B and D do not represent the derivative of $f(x)$. Similarly, note that graph A is negative on the intervals $x < -2$ and $0 < x < 2$ and positive on the intervals $-2 < x < 0$ and $x > 2$; for this reason, it does not represent the derivative of $f(x)$.

18. B

At the points of inflection on the graph of $f(x)$, the graph of $f''(x)$ is equal to zero. The points of inflection of the curve are located at $x = -0.3$ and $x = 2.1$. Therefore, the graph of $f''(x)$ is equal to zero at $x = -0.3$ and $x = 2.1$.

To the left of $x = -0.3$, the graph of $f(x)$ is concave up, and to the right it is concave down. Therefore, the graph of $f''(x)$ is positive to the left of $x = -0.3$ and negative to the right.

To the left of $x = 2.1$, the graph of $f(x)$ is concave down, and to the right it is concave up. Therefore, the graph of $f''(x)$ is negative to the left of $x = 2.1$ and positive to the right.

19. C

The points of inflection occur when $f''(x) = 0$.

Determine the second derivative function.

$$f'(x) = 12x^5 + 15x^4$$
$$f''(x) = 60x^4 + 60x^3$$

Solve for x when $f''(x) = 0$.

$$f''(x) = 60x^4 + 60x^3$$
$$0 = 60x^4 + 60x^3$$
$$0 = 60x^3(x + 1)$$
$$x = 0 \text{ and } x = -1$$

The points of inflection occur at $x = 0$ and $x = -1$.

20. WR

The concavity of a function determines whether the graph of a function lies above or below its tangent line at a given point $x = a$.

- If the second derivative of $f(x)$ is positive at $x = a$ and the graph of $f(x)$ is concave up, the graph of $f(x)$ will lie above its tangent line at $x = a$.
- If the second derivative of $f(x)$ is negative at $x = a$ and the graph of $f(x)$ is concave down, the graph of $f(x)$ will lie below its tangent line at $x = a$.

21. WR

The function $f(x)$ is a continuous polynomial function. Therefore, the function $f'(x)$ is also a continuous polynomial function.

Determine the first derivative of $f(x)$.

$$f(x) = x^{101} + x^{51} + x - 1$$
$$f'(x) = 101x^{100} + 51x^{50} + 1$$

Examining the function $f'(x) = 101x^{100} + 51x^{50} + 1$, there are no values for which $f'(x) = 0$. Also, because of the exponents in $f'(x)$, it follows that $f'(x) > 0$ for all x-values. Therefore, the function $f(x)$ is increasing on $-\infty < x < \infty$.

Since $f(-1) = -4$ and $f(1) = 2$, the graph of $f(x)$ does cross the x-axis (between $x = -1$ and $x = 1$). Therefore, the graph of $f(x)$ can only have one x-intercept.

22. D

When the first derivative graph is positive, the polynomial function is increasing. Similarly, when the first derivative graph is negative, the polynomial function is decreasing. Thus, the polynomial function is increasing on the intervals $-\infty < x < -1$ and $0 < x < 1$ and decreasing on the intervals $-1 < x < 0$ and $1 < x < \infty$.

Local minima occur at points on the function where the first derivative is negative to the left of the point and positive to the right. Therefore, there is a local minimum point at $x = 0$.

Local maxima occur at points on the function where the first derivative is positive to the left of the point and negative to the right. Therefore, there are local maximum points: one occurs at $x = -1$, and the other occurs at $x = 1$.

Practice TEST

23. C

A function $f(x)$ is odd if and only if $f(-x) = -f(x)$. Evaluate each of the given functions to determine which is odd.

Step 1

Evaluate the function $f(x) = x^2 - 3x + 2$, when $f(-x)$.

$$f(x) = x^2 - 3x + 2$$
$$f(-x) = (-x)^2 - 3(-x) + 2$$
$$f(-x) = x^2 + 3x + 2$$

Since $f(-x) \neq -f(x)$, $f(x) = x^2 - 3x + 2$ is not an odd function.

Step 2

Evaluate the function $f(x) = -x^4 - x^3 - x^2$, when $f(-x)$.

$$f(x) = -x^4 - x^3 - x^2$$
$$f(-x) = -(-x)^4 - (-x)^3 - (-x)^2$$
$$f(-x) = -x^4 + x^3 - x^2$$

Since $f(-x) \neq -f(x)$, $f(x) = -x^4 - x^3 - x^2$ is not an odd function.

Step 3

Evaluate the function $f(x) = 5x^5 + 2x^3 - 3x$, when $f(-x)$.

$$f(x) = 5x^5 + 2x^3 - 3x$$
$$f(-x) = 5(-x)^5 + 2(-x)^3 - 3(-x)$$
$$f(-x) = -5x^5 - 2x^3 + 3$$

Since $f(-x) = -f(x)$, $f(x) = 5x^5 + 2x^3 - 3x$ is an odd function.

Step 4

Evaluate the function $f(x) = 3x^3 + 4x^2 - 5x - 1$, when $f(-x)$.

$$f(x) = 3x^3 + 4x^2 - 5x - 1$$
$$f(-x) = 3(-x)^3 + 4(-x)^2 - 5(-x) - 1$$
$$f(-x) = -3x^3 + 4x^2 + 5x - 1$$

Since $f(-x) \neq -f(x)$, $f(x) = 3x^3 + 4x^2 - 5x - 1$ is not an odd function.

24. C

The expression $\frac{dv}{dt}$ means the instantaneous rate of change of v with respect to t. The instantaneous rate of change function of v with respect to t is also defined as the acceleration function.

25. 1.25

Step 1

Determine a function that represents the rate at which the total amount of sand is changing.

Subtract $R(t)$ from $A(t)$ to determine the difference function $D(t)$.

$$D(t) = A(t) - R(t)$$
$$= \frac{30t}{1+2t} - \left(4 + 9\sin\left(\frac{5\pi t}{31}\right)\right)$$
$$= \frac{30t}{1+2t} - 4 - 9\sin\left(\frac{5\pi t}{31}\right)$$

Step 2

Find the rate of change of the total amount of sand on the beach at 4 A.M.

Evaluate $D(t)$ when $t = 4$.

$$D(t) = \frac{30t}{1+2t} - 4 - 9\sin\left(\frac{5\pi t}{31}\right)$$
$$D(4) = \frac{30(4)}{1+2(4)} - 4 - 9\sin\left(\frac{5\pi(4)}{31}\right)$$
$$= \frac{120}{9} - 4 - 9\sin\left(\frac{20\pi}{31}\right)$$
$$= 1.25 \text{ m}^3/\text{h}$$

The rate of change of the total amount of sand on the beach at 4 A.M. is $1.25 \text{ m}^3/\text{h}$.

26. C

Step 1

Determine the first derivative of the depth function.

$$D = 2.1\cos(0.52(t-5)) + 6.8$$
$$D' = -2.1\sin(0.52(t-5))(0.52)$$
$$= -1.092\sin(0.52(t-5))$$

Step 2

Graph the derivative function using a graphing calculator. Note that the graphing calculator must be in radian mode.

Press [Y =], and enter $Y_1 = -1.092\sin(0.52(X-5))$.

Press [WINDOW], and enter X:[0, 24, 2], Y:[−5, 5, 1]. Press [GRAPH] to obtain the graph.

Step 3

Identify the time in the morning at which the rate of change of water depth is decreasing the fastest.

The point at which the rate of change of water depth is decreasing the fastest is the local minimum on the graph of D'.

Identify the local minimum on the graph of D' by pressing [TRACE]. Move the cursor to the first local minimum of the derivative graph, and observe the corresponding x-value.

At approximately 8 A.M., the rate of change of water depth is decreasing the fastest.

27. B

Step 1

Find the first derivative function, $\frac{dB}{dM}$.

$$B = M^2\left(k - \frac{M}{2}\right)$$
$$B = kM^2 - \frac{1}{2}M^3$$
$$\frac{dB}{dM} = 2kM - \frac{3}{2}M^2$$

Step 2

Set $\frac{dB}{dM}$ equal to zero, and solve for M.

$$\frac{dB}{dM} = 2kM - \frac{3}{2}M^2$$
$$0 = 2kM - \frac{3}{2}M^2$$
$$0 = M\left(2k - \frac{3}{2}M\right)$$
$$M = 0$$
$$2k - \frac{3}{2}M = 0$$
$$2k = \frac{3}{2}M$$
$$\frac{4}{3}k = M$$

Step 3

Verify the maximum of the function.

A maximum of a function occurs where its derivative is equal to zero and the function is increasing on the left of the zero and decreasing on the right.

There is no need to consider an interval less than zero as the amount of morphine administered cannot be negative.

Interval	$\frac{dB}{dM}$	***B***
$0 < M < \frac{4}{3}k$	+	Increasing
$\frac{4}{3}k < M < 2k$	–	Decreasing

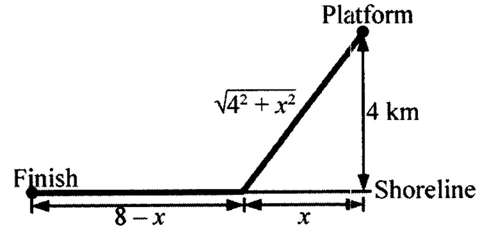
Therefore, the maximum blood pressure will occur when the amount of morphine administered is $\frac{4}{3}k$ mL.

28. WR

Step 1

Sketch a diagram.

The darkened lines in the given diagram illustrate the optimal path for the participant.



Step 2

Define a function for the total time taken by the participant to reach the finish line.

$$t_{\text{swimming}} = \frac{d}{v} = \frac{\sqrt{4^2 + x^2}}{3} = \frac{\sqrt{16 + x^2}}{3}$$

$$t_{\text{running}} = \frac{d}{v} = \frac{8 - x}{5}$$

$$t_{\text{total}} = t_{\text{swimming}} + t_{\text{running}} = \frac{\sqrt{16 + x^2}}{3} + \frac{8 - x}{5} = \frac{1}{3}(16 + x^2)^{\frac{1}{2}} + \frac{1}{5}(8 - x)$$

Step 3

Determine the first derivative of the time function.

$$t = \frac{1}{3}(16 + x^2)^{\frac{1}{2}} + \frac{1}{5}(8 - x)$$

$$\frac{dt}{dx} = \frac{1}{3} \cdot \frac{1}{2}(16 + x^2)^{-\frac{1}{2}}(2x) + \frac{1}{5}(-1) = \frac{x}{3\sqrt{16 + x^2}} - \frac{1}{5}$$

Step 4

Solve for x when $\frac{dt}{dx} = 0$.

$$\frac{dt}{dx} = \frac{x}{3\sqrt{16+x^2}} - \frac{1}{5}$$
$$0 = \frac{x}{3\sqrt{16+x^2}} - \frac{1}{5}$$
$$\left(\frac{1}{5}\right)^2 = \left(\frac{x}{3\sqrt{16+x^2}}\right)^2$$
$$\frac{1}{25} = \frac{x^2}{9(16+x^2)}$$
$$144 + 9x^2 = 25x^2$$
$$0 = 16x^2 - 144$$
$$0 = 16(x^2 - 9)$$
$$x = \pm 3$$

The only possible solution is $x = 3$, since x is a distance. Since $\frac{dt}{dx} < 0$ on $0 < x < 3$, and $\frac{dt}{dx} > 0$ on $3 < t < 8$, time is at a minimum when $x = 3$. Therefore, the participant should exit the water 3 km from the point on the shoreline closest to the water platform.

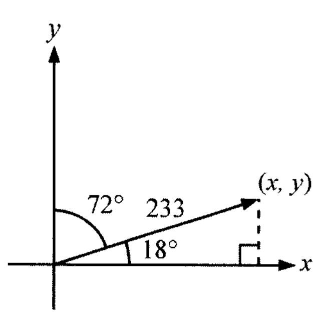
29. C

Vector quantities have both magnitude and direction. A balloon rising vertically at 1.50 km/h is a vector quantity. The magnitude is 1.50 km/h, and the direction is vertically upward.

- A woman's mass of 60 kg is a scalar quantity because the magnitude is 60 kg, but there is no direction.
- A car with a speed of 2.00 m/s is a scalar quantity because it's magnitude is 2.00 m/s, but there is no direction.
- A liquid with a density of 1.0 g/ml is a scalar quantity because it's magnitude is1.0 g/ml, but there is no direction.

30. B

This sketch represents the force vector 233.0 N at a bearing of 72.0°.



The angle formed with respect to the positive x-axis is $90° - 72° = 18°$. Thus, the polar form of the vector is (233, 18°). The Cartesian form, (x, y), of the vector is found by using the formulas $x = r\cos\theta$ and $y = r\sin\theta$.

$$x = r\cos\theta$$
$$= 233\cos 18°$$
$$\approx 221.6$$
$$y = r\sin\theta$$
$$= 233\sin 18°$$
$$\approx 72.0$$

Thus, the Cartesian representation of the vector is (221.6, 72.0).

31. 5

Step 1

Determine the sum $2\vec{p} + 3\vec{q}$ in terms of $\vec{p}$ and $\vec{q}$.

$$2\vec{p} + 3\vec{q}$$
$$= 2(-3, a) + 3(4, 6)$$
$$= (2(-3), 2(a)) + (3(4), 3(6))$$
$$= (-6, 2a) + (12, 18)$$
$$= (-6 + 12, 2a + 18)$$
$$= (6, 2a + 18)$$

Step 2

Solve for a.

$$2\vec{p} + 3\vec{q} = (6, 2a + 18)$$
$$(6, 28) = (6, 2a + 18)$$
$$2a + 18 = 28$$
$$2a = 10$$
$$a = 5$$

32. D

Step 1

Determine the overall force exerted by each team. Let $\vec{x}$ represent the force exerted by the fourth member of team B.

$$\vec{F}_A = 215 + 230 + 250 + 280$$
$$= 975 \text{ N east}$$
$$\vec{F}_B = 3(235) + \vec{x}$$
$$= (705 + \vec{x}) \text{ N west}$$

Step 2

Determine the force, $\vec{x}$, exerted by the fourth member of team B.

$$|\vec{F}_{net}| = |\vec{F}_B| - |\vec{F}_A|$$
$$105 = (705 + x) - 975$$
$$105 = x - 270$$
$$x = 105 + 270$$
$$x = 375 \text{ N}$$

In order for team B to win, the fourth member needs to pull with a force of 375 N west.

33. 11.2

Determine the magnitude of the projection.

$$\begin{aligned}|\text{proj}_{\vec{v}}\vec{u}| &= \frac{|\vec{u}\cdot\vec{v}|}{|\vec{v}|}\\ &= \frac{|(12, 5)\cdot(3, 4)|}{\sqrt{3^2+4^2}}\\ &= \frac{|12(3)+5(4)|}{\sqrt{25}}\\ &= \frac{56}{5}\\ &= 11.2\end{aligned}$$

34. WR

Step 1

Write equations defining a and b using the definition of perpendicular (orthogonal) vectors, where $\vec{u}\cdot\vec{v}=0$.

$$\begin{aligned}\vec{F}_1\cdot\vec{F}_3 &= 0\\ (-5, 20, 25)\cdot(a, b, 45) &= 0\\ -5a+20b+25(45) &= 0\\ -5a+20b &= -1\ 125\\ \vec{F}_2\cdot\vec{F}_3 &= 0\\ (20, 10, -4)\cdot(a, b, 45) &= 0\\ 20a+10b-4(45) &= 0\\ 20a+10b &= 180\end{aligned}$$

Step 2

Solve the system of equations by using the elimination method.

① $-5a+20b = -1\ 125$

② $20a+10b = 180$

Add equation (2) to 4 times equation (1).

$$\begin{aligned}4\times ①\quad -20a+80b &= -4\ 500\\ ②\quad +20a+10b &= 180\\ \hline 90b &= -4\ 320\\ b &= -48\end{aligned}$$

Solve for a by substituting -48 for b into equation (1).

$$\begin{aligned}-5a+20b &= -1\ 125\\ -5a+20(-48) &= -1\ 125\\ -5a-960 &= -1\ 125\\ -5a &= -165\\ a &= 33\end{aligned}$$

For the third force, $a = 33$ and $b = -48$.

35. B

The cross product $\vec{u}\times\vec{v}$ defines an orthogonal vector to both $\vec{u}$ and $\vec{v}$.

36. a) WR

Step 1

Determine the cross product $\vec{u}\times\vec{v}$.

$$\begin{aligned}&\vec{u}\times\vec{v}\\ &= (2, 0, -6)\times(0, 3, -2)\\ &= \begin{pmatrix}[(0)(-2)-(-6)(3)],\\ [(-6)(0)-2(-2)],\\ [(2)(3)-(0)(0)]\end{pmatrix}\\ &= (18, 4, 6)\end{aligned}$$

Step 2

Determine whether vector $\vec{w}$ is orthogonal to the base defined by $\vec{u}$ and $\vec{v}$. For $\vec{w}$ to be orthogonal, it must be a multiple of the cross product $(18, 4, 6)$. Since $\vec{w} = (9, -2, 0)$ is not a multiple, it is not orthogonal to the base of the parallelepiped.

b) WR

Determine the volume of the parallelepiped using the formula $V = |\vec{w}\cdot(\vec{u}\times\vec{v})|$.

$$\begin{aligned}V &= |\vec{w}\cdot(\vec{u}\times\vec{v})|\\ &= |(9, -2, 0)\cdot(18, 4, 6)|\\ &= |9(18)+4(-2)+0(6)|\\ &= |162-8+0|\\ &= 154\text{ units}^3\end{aligned}$$

The volume of the parallelepiped is 154 units^3.

37. WR

Step 1

Write each equation in parametric form.

① $(x, y) = (1, 4)+k(2, -3)$

$x = 1+2k$

$y = 4-3k$

② $(x, y) = (3, 1)+l(9, -3)$

$x = 3+9l$

$y = 1-3l$

Step 2

Equate the x- and y-variables.

$x_1 = x_2$

$1+2k = 3+9l$

③ $2k-9l = 2$

$y_1 = y_2$

$4-3k = 1-3l$

④ $-3k+3l = -3$

Step 3
Solve for k and l using the elimination method for equations (3) and (4).
③ $2k - 9l = 2$
④ $-3k + 3l = -3$
Multiply equation (4) by 3 to form equation (5), and then add it to equation (3).
⑤ $-9k + 9l = -9$
③ $\underline{2k - 9l = 2}$
$-7k + 0l = -7$
$-7k = -7$
$k = 1$
Substitute the value of k into equation (3), and solve for l.
$2k - 9l = 2$
$2(1) - 9l = 2$
$-9l = 0$
$l = 0$
Since k and l are defined by single values, the system of linear equations has one solution, namely the point (3, 1).

38. C

The plane $x + y - 4 = 0$ is parallel to the z-axis, and the plane $x + z - 4 = 0$ is parallel to the y-axis. Thus, these two planes intersect to form a line. Using the elimination method to solve a system of equations supports this finding.

Subtract the equation $x + 0y + z - 4 = 0$ from the equation $x + y + 0z - 4 = 0$.
$x + y + 0z - 4 = 0$
$\underline{x + 0y + z - 4 = 0}$
$0x + y - x = 0$
$y - x = 0$

The equation $y - z = 0$ represents a line.

39. C

Step 1
Determine the parametric form of the equation.
$\vec{r} = (-8, 4) + t(2, -3)$
$= (-8, 4) + (2t, -3t)$
$= (-8 + 2t, 4 - 3t)$
Therefore, the parametric form is as follows:
$x = -8 + 2t$
$y = 4 - 3t$

Step 2
Rewrite each part in terms of t.
$x = -8 + 2t$
$-2t = -x - 8$
$t = \frac{x+8}{2}$
$y = 4 - 3t$
$3t = -y + 4$
$t = \frac{-y+4}{3}$

Step 3
Equate the expressions representing t, and rearrange the result to define the equation in scalar form.
$t = t$
$\frac{x+8}{2} = \frac{-y+4}{3}$
$3(x + 8) = 2(-y + 4)$
$3x + 24 = -2y + 8$
$3x + 2y + 16 = 0$
The scalar form of the line is $3x + 2y + 16 = 0$.

40. A

Step 1
Determine a vector form of the equation of the line passing through the two points.
A position vector could be (3, −2, −6). A direction vector could be determined as follows:
$\vec{PQ} = (2, -1, -3) - (3, -2, -6)$
$= (2 - 3, -1 - (-2), -3 - (-6))$
$= (-1, 1, 3)$
Thus, the vector form of the equation of the line is $\vec{r} = (3, -2, -6) + t(-1, 1, 3)$.

Step 2
Determine the value of t. Since an x-intercept is expressed as $(x, 0, 0)$, set $y = 0$ and solve for t.
$y = -2 + 1t$
$0 = -2 + t$
$t = 2$
Note that the value of t could also have been solved by setting $z = 0$.

Step 3
Substitute 2 for t into the equation $x = 3 - 1t$ to determine the x-intercept, $(x, 0, 0)$.
$x = 3 - 1t$
$x = 3 - 1(2)$
$x = 1$
Therefore, the x-intercept of the line passing through the two points is (1, 0, 0).

41. WR

Step 1
Determine the normal vectors, $\vec{n}$, to each plane.
$\vec{n}_1 = (6, 4, -2)$
$\vec{n}_2 = (15, 10, -5)$
$\vec{n}_3 = (-3, -2, 1)$

Step 2
Determine the configuration of the three planes.
The normal $\vec{n}_2$ is a scalar multiple of $\vec{n}_1$.

$\vec{n}_2 = k\vec{n}_1$
$(15, 10, -5) = k(6, 4, -2)$

$15 = 6k$ $k = \frac{5}{2}$	$10 = 4k$ $k = \frac{5}{2}$	$-5 = -2k$ $k = \frac{5}{2}$

Since $\vec{n}_2 = \frac{5}{2}\vec{n}_1$, planes P_1 and P_2 are parallel.

The normal $\vec{n}_3$ is also a scalar multiple of $\vec{n}_1$.

$\vec{n}_3 = k\vec{n}_1$
$(-3, -2, 1) = k(6, 4, -2)$

$-3 = 6k$ $k = -\frac{1}{2}$	$-2 = 4k$ $k = -\frac{1}{2}$	$1 = -2k$ $k = -\frac{1}{2}$

Since $\vec{n}_3 = -\frac{1}{2}\vec{n}_1$, planes P_1 and P_3 are parallel.

Therefore, the configuration is three parallel planes.

42. B

Since $\vec{n} = (-3, 0, 5)$, the equation in scalar form is $-3x + 0y + 5z + D = 0$. Since $P(6, 3, 1)$ is a point on the plane, substitute the coordinates and solve for D.

$$\begin{aligned} -3x + 0y + 5z + D &= 0 \\ -3(6) + 0(3) + 5(1) + D &= 0 \\ -18 + 5 + D &= 0 \\ D &= 13 \end{aligned}$$

Therefore, the equation of the plane in scalar form is $-3x + 5z + 13 = 0$.

43. A

Step 1
Determine a normal vector to the plane, $\vec{n}$.
$\vec{n} = (3, -4, 2)$

Step 2
Determine a vector in the plane.
Substitute $x = 0$ and $y = 0$ into the equation of the plane to find z in the point $P_1(0, 0, z)$ on the plane.
Then, substitute $x = 0$ and $z = 0$ into the equation of the plane to find y in the point $P_2(0, y, 0)$.

$$\begin{aligned} P_1:\quad 3x - 4y + 2z - 12 &= 0 \\ 3(0) - 4(0) + 2z - 12 &= 0 \\ 2z - 12 &= 0 \\ z &= 6 \end{aligned}$$

$$\begin{aligned} P_2:\quad 3x - 4y + 2z - 12 &= 0 \\ 3(0) - 4y + 2(0) - 12 &= 0 \\ -4y - 12 &= 0 \\ y &= -3 \end{aligned}$$

Thus, the points $P_1(0, 0, 6)$ and $P_2(0, -3, 0)$ lie on the plane. Therefore, vector $\overrightarrow{P_1P_2}$ on the plane is as follows:

$$\begin{aligned} \overrightarrow{P_1P_2} &= (0, -3, 0) - (0, 0, 6) \\ &= (0, -3, -6) \end{aligned}$$

Step 3
Determine a second vector in the plane by taking the cross product of $\vec{n}$ and $\overrightarrow{P_1P_2}$.

$$\begin{aligned} &\vec{n} \times \overrightarrow{P_1P_2} \\ &= (3, -4, 2) \times (0, -3, -6) \\ &= (24 - (-6), 0 - (-18), -9 - 0) \\ &= (30, 18, -9) \end{aligned}$$

Step 4
Determine the parametric form of the plane.
When using the point $P_1(0, 0, 6)$ and the two non-parallel vectors in the plane, the vector form of the equation becomes as follows:
$(x, y, z) = (0, 0, 6) + s(0, -3, -6) + t(30, 18, -9)$
Therefore, the vector form of the plane has the following parameters:
$x = 30t$
$y = -3s + 18t$
$z = 6 - 6s - 9t$

44. WR

Step 1
Determine a vector normal to the plane, $\vec{n}$.
According to the vector equation of the plane, two direction vectors are $\vec{u} = (2, 1, 3)$ and $\vec{v} = (2, -1, -1)$. A vector normal to the plane, $\vec{n}$, is found by taking the cross product of $\vec{u}$ and $\vec{v}$.

$$\begin{aligned} \vec{n} &= \vec{u} \times \vec{v} \\ &= (2, 1, 3) \times (2, -1, -1) \\ &= (-1 - (-3), 6 - (-2), -2 - 2) \\ &= (2, 8, -4) \end{aligned}$$

Step 2

Calculate the dot product between the plane's normal, $\vec{n}$, and a direction vector of the line, such as $\vec{w} = (-4, 2, 2)$.

$$\begin{aligned}\vec{n}\cdot\vec{w} &= (2, 8, -4)\cdot(-4, 2, 2)\\ &= (2)(-4)+(8)(2)+(-4)(2)\\ &= 0\end{aligned}$$

Since the dot product is 0, vector $\vec{w}$ is perpendicular to $\vec{n}$, which means that L_1 and P_1 are parallel.

This could mean that the line lies on the plane or is parallel to the plane.

Step 3

Write the equation of the plane in scalar form.

Since $\vec{n} = (2, 8, -4)$ and a point on the plane is $(2, 1, -2)$, find the scalar form representing the plane, P_1.

$$\begin{aligned}2x+8y-4z+D &= 0\\ 2(2)+8(1)-4(-2)+D &= 0\\ 20+D &= 0\\ D &= -20\end{aligned}$$

The scalar form representing the plane is $P_1: 2x+8y-4z-20=0$.

Step 4

Use a test point on the line to see whether it also lies on the plane.

A point on the line L_1 is $(3, 4, 1)$. Check whether it lies on the plane by substituting the coordinates at the point $(3, 4, 1)$ into the scalar equation of the plane.

$$\begin{aligned}2x+8y-4z-20 &= 0\\ 2(3)+8(4)-4(1)-20 &= 0\\ 6+32-4-20 &= 0\\ 14 &\neq 0\end{aligned}$$

Since the L.H.S $\neq$ R.H.S., the point $(3, 4, 1)$ does not lie on the plane. Therefore, the line L_1 and the plane P_1 are parallel and distinct.

45. WR

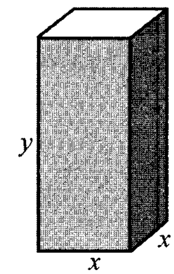
Where the second derivative of a function is negative, the graph of the function will be concave down. Thus, $f(x)$ will be concave down on $a < x < b$.

46. WR

Step 1

Sketch a diagram.

It is given that the surface area of the open-topped box is $SA = x^2 + 4xy$. Since 27 m^2 of material is available, $27 = x^2 + 4xy$.



Step 2

Solve for y in the surface area equation.

$$\begin{aligned}27 &= x^2+4xy\\ 27-x^2 &= 4xy\\ \frac{27}{4x}-\frac{x^2}{4x} &= y\\ \frac{27}{4}x^{-1}-\frac{1}{4}x &= y\end{aligned}$$

Step 3

Substitute the equation for y into the volume function $V = x^2y$.

$$\begin{aligned}V &= x^2y\\ &= x^2\left(\frac{27}{4}x^{-1}-\frac{1}{4}x\right)\\ &= \frac{27}{4}x-\frac{1}{4}x^3\end{aligned}$$

Step 4
Determine the dimensions that yield a maximum volume.
A maximum of a function can occur where its derivative is equal to zero. The maximum occurs when the function is increasing on the left of the zero slope and decreasing on the right.
Determine the first derivative of the volume function, and solve for x when $V' = 0$.

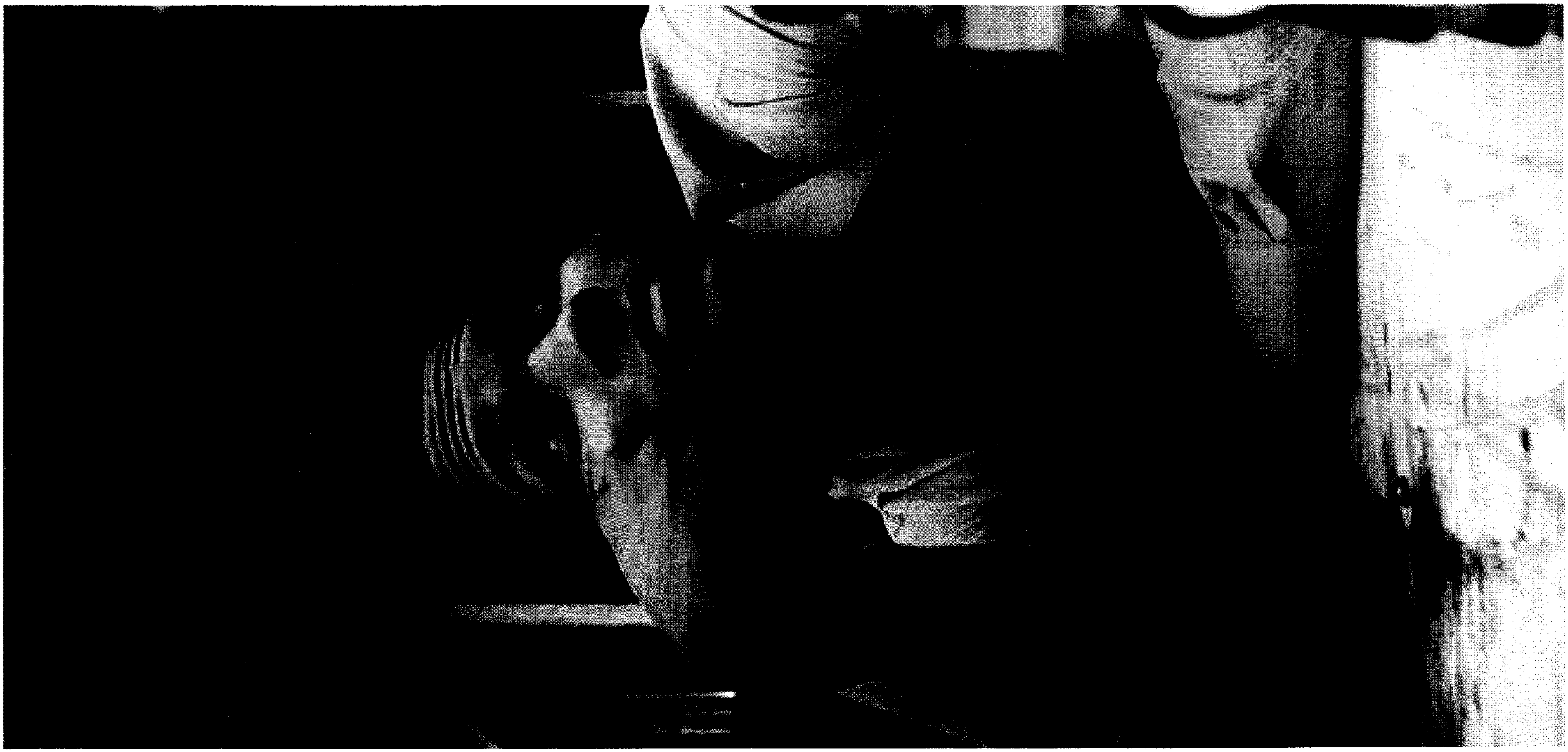
$$V = \frac{27}{4}x - \frac{1}{4}x^3$$
$$V' = \frac{27}{4} - \frac{3}{4}x^2$$
$$0 = \frac{27}{4} - \frac{3}{4}x^2$$
$$0 = \frac{3}{4}(9 - x^2)$$
$$x = \pm 3$$

Since x and y represent dimensions of a box, negative values are not permitted. Thus, only the positive value of x is considered.
When $x = 3$, the volume will be at a maximum.
Solve for y when $x = 3$.

$$\frac{27}{4}x^{-1} - \frac{1}{4}x = y$$
$$\frac{27}{4(3)} - \frac{1}{4}(3) = y$$
$$\frac{27}{12} - \frac{3}{4} = y$$
$$\frac{3}{2} = y$$
$$y = 1.5$$

Therefore, a box with maximum volume will be created when the base has dimensions of 3 m by 3 m with a height of 1.5 m.

Appendices



DATA SHEET

VECTORS

Dot Product

$\vec{u}\cdot\vec{v}=|\vec{u}||\vec{v}|\cos\theta$ $\quad (u_1,u_2)\cdot(v_1,v_2)=u_1v_1+u_2v_2$ $\quad (u_1,u_2,u_3)\cdot(v_1,v_2,v_3)=u_1v_1+u_2v_2+u_3v_3$

Cross Product

$|\vec{u}\times\vec{v}|=|\vec{u}||\vec{v}|\sin\theta$ $\quad (u_1,u_2,u_3)\times(v_1,v_2,v_3)=(u_2v_3-u_3v_2,u_3v_1-u_1v_3,u_1v_2-u_2v_1)$

CALCULUS

First Principles Definitions

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \qquad f'(x_0)=\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

Constant Multiple Rule for Differentiation

$$\frac{d}{dx}kf(x)=kf'(x)$$

Power Rule for Differentiation

$$\frac{d}{dx}\left(x^n\right)=nx^{n-1}$$

Sum and Difference Rule for Differentiation

$$\frac{d}{dx}\left[f(x)\pm g(x)\right]=f'(x)\pm g'(x)$$

Product Rule for Differentiation

$$\frac{d}{dx}\left[f(x)g(x)\right]=f'(x)g(x)+f(x)g'(x)$$

Quotient Rule for Differentiation

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right]=\frac{f'(x)g(x)-f(x)g'(x)}{\left[g(x)\right]^2}$$

Chain Rule

If $y=f(u)$ and $u=g(x)$, then $\frac{dy}{dx}=\frac{dy}{du}\frac{du}{dx}$. If $f(x)=g(h(x))$, then $f'(x)=g'(h(x))h'(x)$

Exponential Functions

$$\frac{d}{dx}e^x=e^x \qquad \frac{d}{dx}a^x=a^x\ln a$$

Trigonometric Functions

$$\frac{d}{d\theta}\sin\theta=\cos\theta \qquad \frac{d}{d\theta}\cos\theta=-\sin\theta \qquad \frac{d}{d\theta}\tan\theta=\frac{1}{\cos^2\theta}$$

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